Université Paris Cité

École Doctorale Sciences Mathématiques de Paris Centre (ED 386)
Institut de Recherche en Informatique Fondamentale

Thèse de Doctorat en Informatique

Signature packing and vertex partition of signed graphs and graphs

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Présentée et soutenue publiquement le 23, Juin 2023
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Résumé

La partition des graphes est l'un des problèmes centraux de la théorie des graphes, issu du célèbre problème des 4 couleurs. Dans cette thèse, deux problèmes majeurs concernant la partition des graphes sont considérés : la séparation des arêtes des graphes signés et la décomposition des sommets des graphes creux.

Dans la première partie de cette thèse, nous nous concentrons sur les problèmes de packing de signature des graphes signés. Un graphe signé (G, σ) est un graphe G équipé d'une signature σ qui attribue à chaque arête de G un signe ($+$ ou $-$). Le concept clé qui sépare un graphe signé d'un graphe 2-arêtes-colorées est la notion de commutation, une commutation sur un sous-ensemble X de sommets de G consiste à multiplier les signes de toutes les arêtes dans la coupe d'arête $(X, V \backslash X)$ par $-$. Un graphe signé (G, σ') est dit équivalent au sens de commutation (ou simplement équivalent) à (G, σ) s'il est obtenu par une commutation sur une coupe d'arête. Ensuite, nous définissons le nombre de packing de signature du graphe signé (G, σ), noté $\rho(G, \sigma)$, comme le nombre maximal de signatures tel que chaque σ_i est équivalent à σ, et les ensembles $E^-_{\sigma_i}$, arêtes négatifs de (G, σ_i), sont disjoints par paires. Nous montrons qu'il est capturé par un homomorphisme spécifique. Et nous établissons une connexion avec plusieurs problèmes bien connus : par exemple, le problème des quatre couleurs, la conjecture de coloration des arêtes de Seymour. Plus précisément, nous montrons d'abord que si G est un graphe simple biparti sans K_5-mineur, alors pour toute signature σ, nous avons $\rho(G, \sigma) \geqslant 4$. Ensuite, nous continuons à utiliser le langage du nombre de packing et étendons la technique pour vérifier que pour tout graphe planaire signé anti-équilibré (G, σ) de circonférence négative d'au moins 5, nous avons $\rho(G, \sigma) \geqslant 5$. Enfin, nous étudions une généralisation du nombre de packing. Au lieu de considérer une signature et ses signatures équivalentes, nous séparons k signatures qui ne sont pas nécessairement équivalentes. Puisqu'il existe un graphe planaire de nombre de packing 1, nous recherchons des conditions suffisantes pour un graphe planaire tel que nous puissions séparer 2 ou 3 signatures.

La deuxième partie de cette thèse porte sur la décomposition des sommets des graphes creux. Nous étudions d'abord la décomposition des sommets des graphes planaires de circonférence au moins 5. Il est connu que tout graphe planaire de circonférence au moins 5 peut être décomposé en deux sous-graphes induits, l'un de degré maximum au plus 3, l'autre de degré maximum au plus 5. Nous renforçons ce résultat en montrant que ces sous-graphes induits peuvent être choisis pour être des forêts. Nous travaillons ensuite sur les graphes creux avec une condition de degré moyen maximum. Plus précisément, nous utilisons la méthode du potentiel pour prouver que tout graphe G de $mad(G) \leqslant \frac{16}{5}$

peut être un sommet décomposé en deux forêts de degré maximum au plus 1 et 4. Par conséquent, tout graphe de genre au plus 1 et de circonférence d'au moins 6 admet également une telle décomposition.

Mots clés: graphe signé, nombre de packing, graphe creux, décomposition des sommets.

Abstract

Graph partition is one of the central problem in graph theory, originated from the famous 4-color problem. In this thesis, two major problems concerning graph partition are considered: the edge separation of signed graphs and the vertex decomposition of sparse graphs.

In the first part of this thesis, we focus on signature packing problems of signed graphs. A signed graph (G, σ) is a graph G equipped with a signature σ which assigns to each edge of G a sign (either $+$ or $-$). The key concept that separates a signed graph from a 2-edge-colored graph is the notion of switching, a switching at a subset X of vertices of G is to multiply the signs of all edges in the edge-cut $(X, V \backslash X)$ by $-$. A signed graph (G, σ') is said to be switching-equivalent (or simply equivalent) to (G, σ) if it is obtained by a switching on an edge-cut. Then we define the signature packing number of a signed graph (G, σ), denoted $\rho(G, \sigma)$, to be the maximum number of signatures $\sigma_1, \sigma_2, \cdots, \sigma_l$ such that each σ_i is switching equivalent to σ and the sets $E^-_{\sigma_i}$, negative edges of (G, σ_i), are pairwise disjoint. We show that it is captured by specific homomorphism. Then we establish connection to several well-known problems: e.g. the four coloring problem, Seymour's edge coloring conjecture. More precisely, we first show that if G is a K_5-minor-free bipartite simple graph, then for any signature σ we have $\rho(G, \sigma) \geqslant 4$. Secondly, we continue using the language of packing number and extend the technique to verify that for any antibalanced signed planar graph (G, σ) of negative girth at least 5, $\rho(G, \sigma) \geqslant 5$. Thirdly, we study a generalization of the packing number. Instead of considering one signature and its equivalent signatures, we separate k signatures which are not necessarily switching equivalent. Since there exists planar graph of packing number 1, we investigate sufficient conditions for a planar graph such that we could separate 2 or 3 signatures.

The second part of this thesis is about vertex decomposition of sparse graphs. We first study the vertex decomposition of planar graphs of girth at least 5. It is known that every planar graph of girth at least 5 can be vertex decomposed into two induced subgraphs, one of maximum degree at most 3, the other of maximum degree at most 5. We strengthen this result by showing that these induced subgraphs can be chosen to be forests. We then work on sparse graphs with maximum average degree condition. More precisely, we use potential method to prove that every graph G of $mad(G) \leqslant \frac{16}{5}$ can be vertex decomposed into two forests of maximum degree at most 1 and 4. Consequently, every graph of genus at most 1 and girth at least 6 also admits such decomposition.

Keywords: signed graph, packing number, sparse graph, vertex decomposition.

Introduction en français

Le théorème des 4-couleurs, l'une des découvertes les plus remarquables de la théorie des graphes, a été présenté pour la première fois comme une question par Francis Guthrie en 1852, qui a essayé de colorer une carte avec quatre couleurs de telle sorte que deux régions adjacentes n'aient pas la même couleur. Le problème, qui était si simplement décrit mais si difficile à prouver, a attiré l'attention de nombreux mathématiciens de l'époque. Après diverses tentatives pendant plus de cent ans, une première preuve complète, assistée par ordinateur, a été réalisée par Kenneth Appel et Wolfgang Haken en 1976. Cependant, la preuve était impossible pour un humain de vérifier à la main. Depuis la première preuve, un algorithme plus efficace avec moins de configurations a été trouvé par Neil Robertson, Daniel P. Sanders, Paul Seymour et Robin Thomas en 1996.

Basé sur le langage des graphes plutôt que sur les cartes, le théorème des 4 couleurs peut être énoncé comme suit : Chaque graphe planaire peut être correctement 4-coloré. Sur la base de cette déclaration, il y a eu de nombreuses reformulations et généralisations, dont certaines ont motivé l'étude de cette thèse. L'une des reformulations équivalentes les plus célèbres du théorème des 4 couleurs proposé par Tait est que les arêtes de chaque graphe cubique planaire sans pont peut être correctement 3-coloré. Plus tard, Paul Seymour a proposé une conjecture plus générale sur la coloration des arêtes des graphes planaires, en disant que tout graphe planaire k-régulier est k-arête colorable si pour chaque ensemble X de nombre impair de sommets, l'arête coupée $(X, V - X)$ est de taille au moins k. Une autre reformulation du théorème des 4 couleurs concernant la décomposition du graphe est qu'un graphe planaire est 4-colorable si et seulement si son ensemble de sommets peut être décomposé en quatre parties, chaque partie induit un ensemble indépendant, ce qui a encore inspiré l'étude de la décomposition des sommets problèmes de graphes.

Dans cette thèse, nous considérons le problème de packing qui sépare l'ensemble des arêtes d'un graphe signé, tel que les sous-ensembles d'arêtes sont des signatures équivalentes du graphe signé, et le problème de décomposition de sommets qui partitionne l'ensemble de sommets d'un graphe creux, de sorte que les sous-ensembles de sommets induisent des graphes spécifiques. Ces deux problèmes capturent le théorème des 4 couleurs. Nous définissons le nombre de packing du graphe signé, et montrons qu'il est capturé par un homomorphisme spécifique. Ensuite, nous établissons une connexion avec plusieurs problèmes bien connus: par exemple, le problème des quatre couleurs, la conjecture de coloration des arêtes de Seymour. Enfin, nous étudions le problème de décomposition des sommets des graphes creux. Plus de détails sont présentés dans les sections suivantes.

La conjecture de coloration des arêtes de Seymour

À la fin du 19e siècle, P. G. Tait a proposé sa propre preuve du théorème des quatre couleurs, bien que la preuve ne soit pas correcte, ses efforts ont abouti à une contribution très importante à la théorie des graphes, puisqu'il a donné une formulation équivalente du théorème des 4 couleurs en termes de coloration des arêtes.

Théorème 1. [47] Tout graphe cubique planaire sans pont est 3-arêtes-colorable.

Notez que ce n'est pas vrai en général pour les graphes cubiques sans pont non planaires, comme le montre le graphe de Petersen. Observez que si un graphe r-régulier est r-arête-colorable, alors chaque classe de couleur est une correspondance parfaite. Par conséquent, pour tout ensemble de sommets X avec un nombre impair de sommets, le nombre d'arêtes qui ont exactement une extrémité dans X (c'est-à-dire la taille de l'arête coupée (X, X^c)) est au moins r. En 1975, P. Seymour a conjecturé qu'avec la condition de planarité, l'affirmation opposée est également vraie.

Conjecture 1. [46] Tout k-graphe planaire est k-arêtes-colorable.

Ici un k-graphe est un multigraphe k-régulier tel que chaque ensemble X de nombre impair de sommets l'arête coupée $(X, V\backslash X)$ est de taille au moins k. La Conjecture 1 a été vérifiée pour les cas de $k \leqslant 8$. Alors que les cas $k = 0, 1, 2$ sont triviaux, le cas $r = 3$ indique que tout graphe planaire cubique sans pont est 3-arêtes-colorable. Par le résultat de Tait, ceci est équivalent au théorème des 4 couleurs. Les cas $k = 4$ et $k = 5$ ont été prouvés par B. Guenin [23] en se basant sur la notion de packing des T-joints. Cependant, l'œuvre de Guenin reste inédit. Le cas suivant $k = 6$ a été résolu par Dvořák, Kawarabayashi et Král' [18] en 2016. La preuve pour le cas $k = 7$ a été donnée par Chudnowsky, Edwards, Kawarabayashi et Seymour [14]. Le cas $k = 8$ a été résolu par Chudnowsky, Edwards et Seymour [15]. Toutes ces preuves pour les valeurs $k \geqslant 4$ sont basées sur des réductions au cas précédent, par conséquent, le théorème des 4 couleurs est supposé. De plus, la preuve des cas $k = 6, 7, 8$ s'appuie sur la preuve non publiée des cas $k = 4, 5$.

Homomorphisme au cube projectif signé

Un cube projectif de dimension d, noté $\mathcal{PC}_d$, est construit à partir d'un hypercube $\mathcal{H}_d$ en ajoutant une nouvelle arête entre chaque paire de sommets antipodaux dans $\mathcal{H}_d$. Notez que K_4 est un cube projectif de dimension 2. Par conséquent, le théorème des 4 couleurs est équivalent à l'affirmation selon laquelle tout graphe planaire (simple) correspond à $\mathcal{PC}_2$. En 2007, R. Naserasr a conjecturé ce qui suit, qui est une généralisation du théorème des 4 couleurs.

Conjecture 2. [34] Tout graphe planaire de circonférence impaire au moins $2d + 1$ admet un homomorphisme à $\mathcal{PC}_{2d}$.

Un cube projectif signé de dimension d, noté $\mathcal{SPC}_d$, est obtenu à partir de $\mathcal{PC}_d$ en attribuant un signe positif à toutes les arêtes de l'hypercube $\mathcal{H}_d$ et un signe négatif aux arêtes entre chaque paire de sommets antipodaux dans $\mathcal{H}_d$. En 2005, B. Guenin a proposé la conjecture suivante.

Conjecture 3. [24] Tout graphe planaire bipartite signé de circonférence négatif $2d$ admet un homomorphisme à $\mathcal{SPC}_{2d-1}$.

Plus tard en 2013, R. Naserasr, E. Rollová et É. Sopena a prouvé que les deux conjectures ci-dessus sont fortement liées à la Conjecture 1 de Seymour sur la coloration des arêtes.

Théorème 2. [34] Tout $(2d + 1)$-graphe planaire est $(2d + 1)$-arête-colorable si et seulement si tout graphe planaire de circonférence impaire au moins $2d + 1$ admet un homomorphisme à $\mathcal{PC}_{2d}$.

Théorème 3. [37] Tout $2d$-graphe planaire est $2d$-arête-colorable si et seulement si tout graphe biparti signé planaire de circonférence déséquilibrée au moins $2d$ admet un homomorphisme à $\mathcal{SPC}_{2d-1}$.

Décomposition des sommets du graphe

Soit $\mathcal{C}_1, \ldots, \mathcal{C}_k$ désignent k classes de graphes. Si $V(G)$ peut être partitionné en k sous-ensembles de sommets $V_1, \ldots, V_k$ tel que le sous-graphe $G[V_i]$ appartienne à $\mathcal{C}_i$ pour chaque $1 \leqslant i \leqslant k$, alors nous appelons une telle partition de sommets une $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$-partition. Pour simplifier, nous utilisons F, F_d, Δ_d et I pour dénoter respectivement la classe des forêts, la classe des forêts de degré maximum au plus d, la classe des graphes de degré maximum au plus d, et la classe des graphes vides. Il est évident que $I = \Delta_0 = F_0$ et $\Delta_1 = F_1$. Le problème des partitions de sommets des graphes sous certaines restrictions sur les conditions de circonférence ou le sparseness a été largement étudié.

Le théorème des 4 couleurs garantit que tout graphe planaire admet une (I, I, I, I)-partition. Notez qu'il existe un graphe planaire, par exemple, le graphe complet K_4, n'ayant aucune (I, I, I)-partition. Le résultat de O. V. Borodine [2] sur la coloration acyclique implique notamment que tout graphe planaire admet une (I, F, F)-partition. C'est le meilleur dans le sens où, comme montré dans [7] par G. Chartrand et H. V. Kronk, il existe des graphes planaires qui n'admettent pas une (F, F)-partition. K. S. Poh [43], en 1990, a montré que tout graphe planaire admet une (F_2, F_2, F_2)-partition.

A. Raspaud et W. Wang [44] ont prouvé que tout graphe planaire sans k-cycles pour un $k \in \{3, 4, 5, 6\}$ fixe admet une (F, F)-partition. En 2013, M. Chen, A. Raspaud et W. Wang [8] ont amélioré ce résultat aux graphes planaires sans triangles sécants. Soit $\mathcal{PG}_g$

la famille des graphes planaires de circonférence au moins g. Il a été prouvé dans [33] qu'il existe un graphe appartenant à $\mathcal{PG}_4$ n'ayant aucune $(\Delta_{d_1}, \Delta_{d_2})$-partition pour une paire d'entiers non négatifs quelconques d_1 et d_2. En 2017, F. Dross, M. Montassier, et A. Pinlou [17] ont montré que tout graphe de $\mathcal{PG}_4$ a une (F, F_5)-partition.

Pour un graphe dans $\mathcal{PG}_5$, O. V. Borodin, et A. N. Glebov [3] ont prouvé qu'il a une (F, F_0)-partition. F. Havet et J. S. Sereni [27] ont prouvé qu'il possède une (Δ_4, Δ_4)-partition, I. Choi et A. Raspaud [12] ont prouvé qu'il possède une (Δ_3, Δ_5)-partition, ces deux résultats ont été améliorés par I. Choi, G. Yu, et X. Zhang [13] en montrant qu'il possède une (Δ_3, Δ_4)-partition. O. V. Borodine et A. V. Kostochka [5] ont prouvé que tout graphe dans $\mathcal{PG}_5$ admet une (Δ_2, Δ_6)-partition. Et I. Choi et al. [11] a montré qu'il possède une (Δ_1, Δ_{10})-partition. De plus, dans [12], I. Choi et A. Raspaud ont posé la question intéressante suivante.

Question 1. Tout graphe de $\mathcal{PG}_5$ possède-t-il une $(\Delta_{d_1}, \Delta_{d_2})$-partition pour tout $d_1 + d_2 \geqslant 8$, $d_2 \geqslant d_1 \geqslant 1$?

Récemment, X. Li, J. Liu, et J. Lv [31] ont montré que la Question 1 est vraie pour le cas $d_1 = 1$ et $d_2 = 9$. Les seuls cas restants vers la Question 1 sont que $d_1 = 1$ et $7 \leqslant d_2 \leqslant 8$.

Pour les graphes de $\mathcal{PG}_6$, un résultat de R. Škrekovski dans [48] implique que tout graphe de $\mathcal{PG}_6$ a une (Δ_3, Δ_3)-partition. Ceci a été amélioré par G. G. Chappell et al. [6] en prouvant que chaque graphe dans $\mathcal{PG}_6$ a une (F_2, F_2)-partition. En considérant des graphes creux, O. V. Borodin et A. V. Kostochka [5] ont obtenu que tout graphe G satisfaisant $mad(G) \leqslant \frac{16}{5}$ admet une (Δ_1, Δ_4)-partition. Il s'ensuit immédiatement que tout graphe dans $\mathcal{PG}_6$ admet une (Δ_1, Δ_4)-partition. Dans la direction opposée, O. V. Borodin et al. [4] ont construit un graphe dans $\mathcal{PG}_6$ qui n'a pas de (F_0, F_d)-partition, où d est un entier non-négatif.

Contributions et organisations

Packing des signatures dans les graphes signés

Dans la partie II, nous nous concentrons sur les problèmes de packing des graphes signés.

Dans le chapitre 3, nous définissons le nombre de packing de signature d'un graphe signé (G, σ), noté $\rho(G, \sigma)$. Tout d'abord en lien avec des développements récents sur la théorie des homomorphismes de graphes signés nous montrons que pour un graphe signé (G, σ), $\rho(G, \sigma) \geqslant d + 1$ si et seulement si (G, σ) admet un homomorphisme à $\mathcal{SPC}_d^o$, où $\mathcal{SPC}_d^o$ est obtenu à partir de $\mathcal{SPC}_d$ en ajoutant une boucle positive à chaque sommet. Dans des cas particuliers, nous avons: I. Un graphe simple G est 4-colorable si et seulement si $\rho(G, -) \geqslant 2$. II. Un graphe biparti signé (G, σ) correspond à SPC_3 si et seulement si $\rho(G, \sigma) \geqslant 3$ notant que SPC_3 est identique à $(K_{4,4}, M)$, c'est le graphe signé sur $K_{4,4}$ où l'ensemble des arêtes négatives forme un appariement parfait. Sur la restriction aux graphes planaires, I est alors une réaffirmation du théorème des 4 couleurs et II est sous-entendu par un travail inédit de B. Guenin. Après un développement plus approfondi de cette théorie du packing dans les graphes signés, nous donnons une preuve indépendante de II qui fonctionne sur la classe plus large des graphes sans K_5-mineur. Plus précisément, nous prouvons que: Si G est un graphe simple biparti sans K_5-mineur, alors pour toute signature σ, on a $\rho(G,\sigma) \geqslant 4$. L'énoncé s'avère strictement plus fort que le théorème des quatre couleurs et est prouvé en l'assumant. De plus, nous montrons que I ne peut pas être étendu à la classe de tous les graphes simples planaires signés. D'autres développements, y compris les implications algorithmiques, sont envisagés.

Dans le Chapitre 4, nous continuons à utiliser le langage du nombre de packing et nous étendons la technique pour vérifier le cas $k = 5$ de la Conjecture 1. Plus précisément, nous prouvons que pour tout graphe planaire signé anti-équilibré (c'est-à-dire équivalente à tous les bords négatifs) (G,σ) de circonférence négative au moins 5, nous avons $\rho(G,\sigma) \geqslant 5$. Comme preuve du cas $k = 4$, nous donnons d'abord une reformulation du théorème, puis nous faisons l'induction et utilisons différentes déclarations pour différentes directions de l'induction.

Dans le Chapitre 5, nous étudions une généralisation du numéro de packing. Au lieu de considérer une signature et ses signatures équivalentes, nous considérons k signatures $\sigma_1, \sigma_2, \ldots, \sigma_k$ (pas nécessairement équivalentes) et demandons s'il existe des signatures $\sigma'_1, \sigma'_2, \ldots, \sigma'_k$, où σ'_i est une commutation de σ_i, telle que les ensembles d'arêtes négatives $E^-_{\sigma'_i}$ sont disjoints par paires. Il est connu qu'il existe un graphe simple planaire signé dont le nombre de packing est 1 [28]. Ainsi, pour un graphe planaire général, séparer deux signatures n'est pas toujours possible même si $\sigma_1 = \sigma_2$. Dans ce chapitre, nous prouvons qu'étant donné un graphe planaire G sans 4-cycle et deux signatures quelconques σ et π sur G, il existe des commutations σ' et π' de σ et π, respectivement, telles que $E^-_{\sigma'} \cap E^-_{\pi'} = \varnothing$. Comme corollaire de la 3-dégénérescence, nous pourrions également séparer deux signatures sur un graphe planaire sans triangle, ou sans 5-cycle ou sans 6-cycle. De plus, nous prouvons que l'on pourrait séparer trois signatures sur des graphes de degré moyen maximal inférieur à 3, en particulier sur des graphes planaires de circonférence au moins égale à 6.

Décomposition des sommets des graphes creux

Dans la partie III, nous nous concentrerons sur la décomposition des sommets des graphes creux.

Dans le chapitre 6, nous étudions la décomposition des sommets des graphes planaires de circonférence au moins égale à 5. On sait que tout graphe planaire de circonférence au moins 5 admet une (Δ_3, Δ_5)-partition. Dans ce chapitre, nous renforçons ce résultat en prouvant que tout graphe planaire de circonférence au moins 5 admet une (F_3, F_5)-partition.

Dans le chapitre 7, nous étudions la décomposition des sommets des graphes creux de condition de degré moyen maximum. Plus précisément, nous utilisons la méthode des potentiels pour montrer que tout graphe G avec $mad(G) \leqslant \frac{16}{5}$ admet une (F_1, F_4)-partition. En corollaire, tout graphe de faible genre et de circonférence au moins égale à 6 admet une (F_1, F_4)-partition. Nous savons qu'il existe un graphe planaire de circonférence 6 qui n'a pas de (F_0, F_d)-partition [4], où d peut être un entier non négatif.

Ce fait garantit que l'indice de F_1 ne peut pas être amélioré davantage. Pourtant, on ne sait pas si la classe de F_4 peut être renforcée ou non.

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Part I

Introduction

Chapter 1

Introduction

The 4-color theorem, one of the most outstanding discovery in graph theory, was first presented as a question by Francis Guthrie in 1852, who tried to color a map with four colors such that no two adjacent regions have the same color. The problem, which was so simply described but so difficult to prove, caught a lot of attention of many mathematicians at the time. After various attempts during more than one hundred years, a first complete proof, assisted by computer, was achieved by Kenneth Appel and Wolfgang Haken in 1976. However, the proof was infeasible for a human to check by hand. Since the first proof, a more efficient algorithm with less configurations have been found by Neil Robertson, Daniel P. Sanders, Paul Seymour, and Robin Thomas in 1996.

Based on language of graphs rather than maps, the 4-color theorem can be stated as: Every planar graph can be properly 4-colored. Based on this statement, there have been many restatements and generalizations, some of which has motivated the study of this thesis. One of the most famous equivalent restatement of the 4-color theorem proposed by Tait is that every planar bridgeless cubic graph is 3-edge-colorable. Later Paul Seymour proposed a more general conjecture about the edge coloring of planar graphs, saying that every k-regular planar graph is k-edge colorable if for each set X of odd number of vertices the edge cut $(X, V - X)$ is of size at least k. Another restatement of the 4-color theorem regarding graph decomposition is that a planar graph is 4-colorable if and only if its vertex set can be decomposed into four parts, each part induces an independent set, which further inspired the study of the vertex decomposition problems of graphs.

In this thesis, we consider the packing problem which is separating the edge set of a signed graph, such that the edge subsets are equivalent signatures of the signed graph, and vertex decomposition problem which is partitioning the vertex set of a sparse graph, such that the vertex subsets induce specific graphs. Both of these problems capture the 4-color theorem. We define the packing number of signed graph, and show that it is captured by specific homomorphism. And then we establish connection to several well-known problems: e.g. the 4-coloring problem, and Seymour's edge coloring conjecture, etc. Finally, we study the vertex decomposition problem of sparse graphs. More details are shown in the following sections.

1.1 Seymour's edge-coloring conjecture

In the late 19th century, P. G. Tait proposed his own proof of the Four Color Theorem, although the proof was not correct, his efforts resulted in a very important contribution to graph theory, as it gave an equivalent formulation of the 4-color theorem in terms of edge-coloring.

Theorem 1.1. *[47] Every planar bridgeless cubic graph is 3-edge-colorable.*

Note that this is not true in general for non-planar cubic bridgeless graphs, as shown by the Petersen graph. Observe that if an r-regular graph is r-edge-colorable, then every color class is a perfect matching. Therefore, for any vertex set X with odd number of vertices, the number of edges which has exactly one endpoint in X (i.e. the size of the edge cut (X, X^c)) is at least r. In 1975, P. Seymour conjectured that with the condition of planarity the opposite statement is also true.

Conjecture 1.1. *[46] Every planar k-graph is k-edge-colorable.*

Here a k-graph is a k-regular multigraph such that each set X of odd number of vertices the edge cut $(X, V\backslash X)$ is of size at least k. Conjecture 1.1 has been verified for the cases of $k \leqslant 8$. While the cases $k = 0, 1, 2$ are trivial, the case $r = 3$ states that every bridgeless cubic planar graph is 3-edge-colorable. By the result of Tait, this is equivalent to the 4-color theorem. The case $k = 4$ and $k = 5$ were proved by B. Guenin [23] based on the notion of packing T-joins. However, the work of Guenin remains unpublished. The next case $k = 6$ was solved by Dvořák, Kawarabayashi, and Král' [18] in 2016. The proof for the case $k = 7$ was given by Chudnowsky, Edwards, Kawarabayashi, and Seymour [14]. The case $k = 8$ was solved by Chudnowsky, Edwards, and Seymour [15]. All these proofs for the values $k \geqslant 4$ are based on reductions to the previous case, therefore, the 4-color theorem is assumed. Furthermore, the proof of cases $k = 6, 7, 8$ relies on the unpublished proof of the cases $k = 4, 5$.

1.2 Homomorphism to signed projective cube

A *projective cube of dimension* d, denoted by $\mathcal{PC}_d$, is built from a hypercube $\mathcal{H}_d$ by adding a new edge between each pair of antipodal vertices in $\mathcal{H}_d$. Note that K_4 is projective cube of dimension 2. Therefore, 4-color theorem is equivalent to stating that every planar (simple) graph maps to $\mathcal{PC}_2$. In 2007, R. Naserasr conjectured the following, which is a generalization of the 4-color theorem.

Conjecture 1.2. *[34] Every planar graph of odd-girth at least* $2d + 1$ *admits a homomorphism to* $\mathcal{PC}_{2d}$.

A signed projective cube of dimension d, denoted by $\mathcal{SPC}_d$, is obtained from $\mathcal{PC}_d$ by assigning positive sign to all the edges of the hypercube $\mathcal{H}_d$ and negative sign to the edges between each pair of antipodal vertices in $\mathcal{H}_d$. In 2005, B. Guenin proposed the following conjecture.

Conjecture 1.3. *[24] Every signed bipartite planar graph of negative-girth* $2d$ *admits a homomorphism to* $\mathcal{SPC}_{2d-1}$.

Later in 2013, R. Naserasr, E. Rollová and É. Sopena proved that both the above two conjectures are strongly connected to Conjecture 1.1 by Seymour about edge-coloring.

Theorem 1.2. *[34] Every planar* $(2d+1)$*-graph is* $(2d+1)$*-edge-colorable if and only if every planar graph of odd girth at least* $2d+1$ *admits a homomorphism to* $\mathcal{PC}_{2d}$.

Theorem 1.3. *[37] Every planar* $2d$*-graph is* $2d$*-edge-colorable if and only if every planar signed bipartite graph of unbalanced girth at least* $2d$ *admits a homomorphism to* $\mathcal{SPC}_{2d-1}$.

1.3 Vertex decomposition of graphs

Let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ denote k classes of graphs. If $V(G)$ can be partitioned into k vertex subsets $V_1, \ldots, V_k$ such that the subgraph $G[V_i]$ belongs to $\mathcal{C}_i$ for each $1 \leqslant i \leqslant k$, then we call such a vertex partition a $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$-partition. For simplicity, we use F, F_d, Δ_d and I to denote the class of forests, the class of forests of maximum degree at most d, the class of graphs of maximum degree at most d, and the class of empty graphs, respectively. It is obvious that $I = \Delta_0 = F_0$ and $\Delta_1 = F_1$. The problem of vertex partitions of graphs under some restrictions on girth conditions or sparseness has been widely studied.

The 4-color theorem ensures that every planar graph admits an (I, I, I, I)-partition. Note that there exists planar graph, for example, the complete graph K_4, having no (I, I, I)-partition. O. V. Borodin's result [2] on acyclic coloring in particular implies that every planar graph admits an (I, F, F)-partition. This is the best in the sense that, as shown in [7] by G. Chartrand and H. V. Kronk, there are planar graphs which don't admit an (F, F)-partition. K. S. Poh [43], in 1990, showed that every planar graph admits an (F_2, F_2, F_2)-partition.

A. Raspaud and W. Wang [44] proved that every planar graph without k-cycles for some fixed $k \in \{3, 4, 5, 6\}$ admits an (F, F)-partition. In 2013, M. Chen, A. Raspaud and W. Wang [8] improved this result to planar graphs without intersecting triangles. Let $\mathcal{PG}_g$ denote the family of planar graphs of girth at least g. It has been proved in [33] that there is a graph belonging to $\mathcal{PG}_4$ having no $(\Delta_{d_1}, \Delta_{d_2})$-partition for pair of any non-negative integers d_1 and d_2. In 2017, F. Dross, M. Montassier, and A. Pinlou [17] showed that every graph in $\mathcal{PG}_4$ has an (F, F_5)-partition.

For a graph in $\mathcal{PG}_5$, O. V. Borodin, and A. N. Glebov [3] proved it has an (F, F_0)-partition. F. Havet and J. S. Sereni [27] proved that it has a (Δ_4, Δ_4)-partition, and I. Choi and A. Raspaud [12] proved it has a (Δ_3, Δ_5)-partition, these two results have been improved by I. Choi, G. Yu and X. Zhang [13] by showing that it has a (Δ_3, Δ_4)-partition. O. V. Borodin and A. V. Kostochka [5] proved that every graph in $\mathcal{PG}_5$ has a (Δ_2, Δ_6)-partition. And I. Choi et al. [11] proved it has a (Δ_1, Δ_{10})-partition. Moreover, in [12], I. Choi and A. Raspaud put forward the following interesting question.

Question 1.1. Does every graph in $\mathcal{PG}_5$ have a $(\Delta_{d_1}, \Delta_{d_2})$-partition for all $d_1 + d_2 \geqslant 8$, where $d_2 \geqslant d_1 \geqslant 1$?

Recently, X. Li, J. Liu and J. Lv [31] showed that Question 1.1 is true for the case that $d_1 = 1$ and $d_2 = 9$. The only remaining cases toward Question 1.1 are that $d_1 = 1$ and $7 \leqslant d_2 \leqslant 8$.

For graphs in $\mathcal{PG}_6$, a result of R. Škrekovski in [48] implies that every graph in $\mathcal{PG}_6$ has a (Δ_3, Δ_3)-partition. This was further improved by G. G. Chappell et al. [6] by proving that every graph in $\mathcal{PG}_6$ has an (F_2, F_2)-partition. By considering sparse graphs, O. V. Borodin and A. V. Kostochka [5] obtained that every graph G satisfying $mad(G) \leqslant \frac{16}{5}$ admits a (Δ_1, Δ_4)-partition. It follows immediately that every graph in $\mathcal{PG}_6$ admits a (Δ_1, Δ_4)-partition. In the opposite direction, O. V. Borodin et al. [4] constructed a graph in $\mathcal{PG}_6$ which has no (F_0, F_d)-partition, where d is a non-negative integer.

1.4 Contributions and organizations

1.4.1 Packing signatures in signed graphs

In Part II, we focus on packing problems of signed graphs.

In Chapter 3, we define the signature packing number of a signed graph (G, σ), denoted by $\rho(G, \sigma)$. First in connection to recent developments on the theory of homomorphisms of signed graphs we prove that for a signed graph (G, σ), $\rho(G, \sigma) \geqslant d+1$ if and only if (G, σ) admits a homomorphism to $\mathcal{SPC}_d^o$, where $\mathcal{SPC}_d^o$ is obtained from $\mathcal{SPC}_d$ by adding a positive loop to every vertex. In special cases we have: I. A simple graph G is 4-colorable if and only if $\rho(G, -) \geqslant 2$. II. A signed bipartite graph (G, σ) maps to SPC_3 if and only if $\rho(G, \sigma) \geqslant 3$ noting that SPC_3 is the same as $(K_{4,4}, M)$, that is the signed graph on $K_{4,4}$ where the set of negative edges forms a perfect matching. On restriction to planar graphs, I is then a restatement of the 4-color theorem and II is implied by an unpublished work of B. Guenin. After further development of this theory of packing in signed graphs, we give an independent proof of II which works on the larger class of K_5-minor-free graphs. More precisely we prove that: If G is a K_5-minor-free bipartite simple graph, then for any signature σ we have $\rho(G, \sigma) \geqslant 4$. The statement is shown to be strictly stronger than the four-color theorem and is proved assuming it. Furthermore, we show that I cannot be extended to the class of all signed planar simple graphs. Further development, including algorithmic implications, are considered.

In Chapter 4, we continue using the language of packing number and extend the technique to verify the case $k = 5$ of Conjecture 1.1. More precisely, we prove that for any antibalanced (i.e. switching equivalent to all edges negative) signed planar graph (G, σ) of negative girth at least 5, we have $\rho(G, \sigma) \geqslant 5$. As the proof of case $k = 4$, we first provide a reformulation of the theorem, then we do induction and use different statements for different directions of the induction.

In Chapter 5, we study a generalization of the packing number. Instead of considering one signature and its equivalent signatures, we consider k signatures $\sigma_1, \sigma_2, \ldots, \sigma_k$ (not necessarily switching equivalent) and ask whether there exist signatures $\sigma'_1, \sigma'_2, \ldots, \sigma'_k$, where σ'_i is a switching of σ_i, such that the sets of negative edges $E^-_{\sigma'_i}$ are pairwise disjoint. It is known that there exists a signed planar simple graph whose packing number is 1 [28]. Thus for a general planar graph separating two signatures is not always possible even if $\sigma_1 = \sigma_2$. In this Chapter, we prove that given a planar graph G with no 4-cycle and any two signatures σ and π on G, there are switchings σ' and π' of σ and π, respectively, such that $E^-_{\sigma'} \cap E^-_{\pi'} = \varnothing$. As a corollary of 3-degeneracy, we could also separate two signatures on a planar graph with no triangle, or with no 5-cycle or with no 6-cycle. Moreover, we prove that one could separate three signatures on graphs of maximum average degree less than 3, in particular on planar graphs of girth at least 6.

1.4.2 Vertex decomposition of sparse graphs

In Part III, we will focus on the vertex decomposition of sparse graphs.

In Chapter 6, we study the vertex decomposition of planar graphs of girth at least 5. It is known that every planar graph of girth at least 5 admits a (Δ_3, Δ_5)-partition. In this chapter, we strengthen this result by proving that every planar graph of girth at least 5 admits an (F_3, F_5)-partition.

In Chapter 7, we study the vertex decomposition of sparse graphs of maximum average degree condition. More precisely, we use potential method to prove that every graph G with $mad(G) \leqslant \frac{16}{5}$ admits an (F_1, F_4)-partition. As a corollary, every graph with low genus and girth at least 6 admits an (F_1, F_4)-partition. We know that there exists planar graph of girth 6 which has no (F_0, F_d)-partition [4], where d can be any non-negative integer. This fact guarantees that the subscript of F_1 cannot be further improved. Still, whether the class of F_4 can be strengthened or not is unknown.

Chapter 2

Preliminary

A *graph* is a pair $G = (V, E)$, where V is a set whose elements are called *vertices*, and E is a set of paired vertices, whose elements are called *edges*. The vertices of an edge are called the *endpoints* of the edge. An edge having two identical endpoints is called *loop*. If two edges have the same endpoints, then they are *multiedges*. A *simple* graph is a graph without loops and multiedges and a *multigraph* is a graph without loops. The *order* of a graph is its number of vertices $|V|$. The *size* of a graph is its number of edges $|E|$. A *subgraph* H of a graph G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, we say H is a *proper* subgraph of G is either $V(H) \subsetneq V(G)$ or $E(H) \subsetneq E(G)$. For $S \subseteq V$, we say that $G[S]$ is an *induced* subgraph of G which is formed from the vertex set S and all of the edges connecting pairs of S. For a vertex subset X of $V(G)$, the *edge-cut*, denoted by $(X, V\backslash X)$, is the set of edges that have one endpoint in X and have the other endpoint in $V\backslash X$.

A *planar graph* is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. A *face* of the graph is a region bounded by a set of edges and vertices in the embedding. We use $F(G)$ to denote its face set. The *degree* of a vertex v is the number of edges that are incident to v, denoted by $d(v)$. The *degree* of a face f, denoted by $d(f)$, is the number of edges incident with f (a cut-edge is counted twice). The set of *neighbours* of a vertex, denoted by $N(v)$, are all the vertices adjacent to v. A *k-vertex* (resp. *k^+-vertex* and *k^--vertex*) is a vertex of degree k (resp. at least k and at most k). The same notation can be applied to faces. Given a graph G, the *minimum degree* and *maximum degree*, denoted by $\delta(G)$ and $\Delta(G)$, respectively, are the minimum and maximum among all the $d(v)$'s for $v \in V(G)$. The *average degree* of a graph G is defined to be $\frac{2|E|}{|V|}$ and the *maximum average degree* of G, denoted by $mad(G)$, is defined to be $mad(G) = max\{\frac{2|E|}{|V|} : H \subseteq G\}$.

A *walk* of graph G is a sequence of vertices and edges, $v_1, e_1, v_2, \ldots, e_{k-1}, v_k$, such that $1 \leqslant i \leqslant k$, the edge e_i has endpoints v_i and v_{i+1}, note that vertices and edges can be repeated. A walk is said to be a *closed* walk if the starting and ending vertices are identical. A *path* is a walk such that all the vertices are distinct. A *cycle* is a close walk such that all the vertices are distinct except the starting and ending vertices. The *length* of a walk, path or cycle is the number of its edges. A path or a cycle of length k is call a *k-path* or *k-cycle*. An *odd* (or *even*) cycle is a cycle of odd (or even) length. The *girth* of a graph G is defined to be the length of a shortest cycle in G, denote by $g(G)$.

2.1 Signed graph and Homomorphism

Signed graphs were introduced by Harary [25] in 1954. Since then, interest in them has continued to grow and investigations have branched out in many different directions. One consistent theme involves considering definitions and theorems concerning ordinary graphs and seeing how they generalize into the broader class of signed graphs. Sometimes the generalizations can be quite significant and interesting. (Think about how significant and beautiful is the generalization of calculus from $\mathbb{R}$ to $\mathbb{C}$.)

A *signed graph* (G, σ) is a graph G equipped with a *signature* σ which assigns to each edge of G a sign (either $+$ or $-$). When the signature is clear from the context or can be omitted, we sometimes denote the signed graph by $\hat{G}$. An edge with the sign $-$ is called a *negative* edge and an edge with the sign $+$ is called a *positive* edge. Given a signed graph (G, σ), the sets of positive and negative edge of (G, σ) is denoted by E^+_σ and E^-_σ, respectively. Given a labeled signature such as σ_i in (G, σ_i) and when there is no ambiguity, we may write E^-_i in place of $E^-_{\sigma_i}$. Given a signed graph (G, σ) with $E' = E^-_\sigma$, sometime we rather write (G, E') instead of (G, σ). A signed multigraph on two vertices with two parallel edges of different signs is call a *digon*.

With $\{+, -\}$ viewed as a multiplicative group, the key concept that separates a signed graph from a 2-edge-colored graph is the notion of *switching* (also referred to as "resigning" by some researchers). A switching at a vertex v, is to multiply the sign of all edges incident to v by a $-$, noting that a loop on v is incident to it from both ends and, therefore, a switching at v does not change sign of a loop at v. To switch at each of the vertices of a subset X of vertices of G is to multiply the signs of all the edges in the edge-cut $(X, V \backslash X)$ by $-$. A signed graph (G, σ') is said to be *switching-equivalent* (or simply *equivalent*) to (G, σ) if it is obtained from the other by a sequence of switchings or, equivalently, by a switching on an edge-cut. It is easily observed that (G, σ_1) and (G, σ_2) are switching equivalent if and only if the symmetric difference $E^-_1 \triangle E^-_2$ is an edge cut of G. Besides, it is straightforward to check that this is indeed an equivalence relation among all possible signatures.

Given a signed graph (G, σ) and a subgraph H of G, (H, σ) is said to be a signed graph which keeps the sign of the edges as (G, σ). Moreover, if H is a spanning subgraph of G and we get (H, σ') from (H, σ) by switching at $X \subseteq V(H)$, then we say (G, σ') is obtained from (G, σ) by switching at $X \subseteq V(G)$ since H is a spanning subgraph of G. One may easily observe that (H, σ') is also a signed graph that keeps the sign of the edges as (G, σ').

The *Sign* of a structure in (G, σ) is the product of the signs of its edges, considering multiplicity. Structures of highest importance are cycles and closed walks. Note that the sign of either of them is invariant under a switching operation and they determine some crucial properties of a signed graph. An *unbalanced* or *negative* cycle (*balanced* or *positive*) in signed graph is a cycle having an odd (even) number of negative edges.

If every cycle in a signed graph (G, σ) is positive, then (G, σ) is said to be *balanced*. A signed graph (G, σ) is said to be *antibalanced* if $(G, -\sigma)$ is *balanced*. A theorem of Zaslavsky says that the set of negative cycles (equivalently the set of positive cycles) uniquely determines the equivalent class of signatures:

Theorem 2.1. *[51] Given two signatures σ and σ' on a graph G, they are equivalent if and only if they have the same set of negative cycles.*

Closed walks are the key structures of a signed graph. Sign and parity of the length of closed walks partition them into four categories: positive and even, positive and odd, negative and even, negative and odd. Given a signed graph (G, σ), the length of a shortest closed walk in each of these categories will be denoted, respectively, by $g_{00}(G, \sigma)$, $g_{01}(G, \sigma)$, $g_{10}(G, \sigma)$, $g_{11}(G, \sigma)$ the logic being that the first index represents the parity of the number of negative edges and the second represents the parity of the total number of edges. Furthermore, the length of a shortest negative closed walk will be denoted by $g_-(G, \sigma)$ (i.e., $g_-(G, \sigma) = \min\{g_{10}(G, \sigma), g_{11}(G, \sigma)\}$). For each of these parameters, when there is no closed walk of the type that is considered, the corresponding parameter is set to be ∞.

As long as (G, σ) has at least one edge, $g_{00}(G, \sigma)$ is 2 as a traversing an edge in both direction is always a positive closed walk of length 2. It is not difficult to build an example of signed graph (G, σ) where the value of $g_{ij}(G, \sigma)$ for $ij \in \mathbb{Z}_2^2$, $ij \neq 00$ is obtained by a closed walk which is not a cycle. However, at least two of these values, if they are all bounded, will always be obtained by a proper cycle. More precisely, the two smallest of the values $\{g_{01}(G, \sigma), g_{01}(G, \sigma), g_{01}(G, \sigma)\}$ correspond to cycles because if a shortest closed walk of type ij, $ij \in \{01, 10, 11\}$, is not a cycle, it must be formed of merging of the two closed walks of types $\{01, 10, 11\} - ij$. Thus the only value of g_{ij} which is possibly not recognized by a cycle is the largest of the three values. This, in particular, implies that a shortest negative closed walk is always a cycle. Thus $g_-(G, \sigma)$ may also be defined as the length of a shortest negative cycle and referred to as the *negative girth* of (G, σ). We note that in these definitions a loop is considered as a cycle of length 1 and two parallel edges form a cycle of length 2.

Given a graph G, the signed graph $(G, -)$ (respectively, $(G, +)$) is the signed graph where all edges are negative (positive). For a positive integer l, C_{-l} is a negative cycle of length l together with any of its equivalent signatures.We may then denote a positive cycle of length l by C_{+l} or simply by C_l.

Given $ij \in \mathbb{Z}_2^2$, $ij \neq 00$, the class $\mathcal{G}_{ij}$ of signed graphs is defined as follows:

$$\mathcal{G}_{ij} = \{(G, \sigma) \mid g_{i'j'}(G, \sigma) = \infty \text{ for } i'j' \in \mathbb{Z}_2^2 - 00, i'j' \neq ij\}.$$

In other words, given a signed graph $(G, \sigma) \in \mathcal{G}_{ij}$, every closed walk of (G, σ) is either a positive even closed walk or a closed walk whose parity of number of negative edges and the length are determined by i and j, respectively. Thus, based on Theorem 2.1 we have:

- $\mathcal{G}_{01}$ is the class of signed graphs (G, σ) which can be switched to $(G, +)$,

- $\mathcal{G}_{11}$ consists of signed graphs (G, σ) which can be switched to $(G, -)$,
- $\mathcal{G}_{10}$ is the class of all signed bipartite graphs.

Each of the first two items can be regarded as a natural embedding of graphs into the larger class of signed graphs. As $\rho(G, +) = \infty$, the preferred embedding of graphs into signed graphs in the study of packing signatures is $(G, -)$. This has extra advantage that works better with minor theory of signed graphs. The class $\mathcal{G}_{10}$ is also of importance for this study.

2.1.1 Coloring of signed graphs

The concept of coloring of signed graph was first introduced by Zaslavsky [50] in 1981, which is a natural extension and generalization of vertex coloring of graphs. One of the most natural notion is 0-free coloring. Given a signed graph (G, σ) and a positive integer k, a *0-free $2k$-coloring* of (G, σ) is a mapping $c : V(G) \to \{\pm 1, \pm 2, \dots, \pm k\}$ such that for any edge $e = uv$ of (G, σ), $c(u) \neq \sigma(e)c(v)$.

One can easily observe that if (G, σ) contains a positive loop, then it does not admit any proper $2k$-coloring for any k. Furthermore, a signed graph (G, σ) admits a 0-free $2k$-coloring if and only if for every switching equivalent signature σ', (G, σ') also admits 0-free $2k$-coloring.

2.1.2 Homomorphism of signed graphs

Homomorphisms of graphs is an important topic within graph theory and its generalization to signed graphs hints at an even richer theory.

Given signed graphs (G, σ) and (H, π), a *homomorphism* of (G, σ) to (H, π) is a mapping φ of the vertices and edges of G to the vertices and edges of H, respectively, such that adjacencies, incidences and signs of closed walks are preserved. Essentially, regarding Theorem 2.1, a homomorphism is expected to preserve the signs of cycles, however, the image of a cycle could be a closed walk rather than a cycle. One should note that replacing cycles with closed walks in Theorem 2.1 we still have the same conclusion.

When there exists a homomorphism (G, σ) to (H, π) we write $(G, \sigma) \to (H, \pi)$. A homomorphism of (G, σ) to (H, π) is said to be *edge-sign-preserving* if, furthermore, signs of the edges are preserved. When it is needed to distinguish the two notions, the former might be referred to as *switching homomorphism* because of the following connection:

Theorem 2.2. *[39] A signed graph (G, σ) admits a homomorphism to a signed graph (H, π) if for a signature σ' on G, equivalent to σ, the signed graph (G, σ') admits an edge-sign-preserving homomorphism to (H, π).*

The definition of homomorphism implies a basic no-homomorphism lemma:

Lemma 2.1. *If $(G, \sigma) \to (H, \pi)$, then $g_{ij}(G, \sigma) \geqslant g_{ij}(H, \pi)$ for every $ij \in \mathbb{Z}_2^2$.*

It follows from the definitions and Theorem 2.2 that homomorphisms of signed graphs generalize the notion of chromatic number of graphs. More precisely, we have the following observation.

Observation 2.1. *Given a graph G, we have $\chi(G) \leqslant k$ if and only if $(G, -) \to (K_k, -)$.*

This restatement of k-coloring is also helpful to state the Odd-Hadwiger conjecture of Gerards and Seymour (see for example [21]). Recall that minor of a signed graph (G, σ) is a signed graph obtained from (G, σ) by the following four operations: deleting vertices, deleting edges, contracting positive edges, and switching.

Conjecture 2.1 (Odd-Hadwiger)**.** *If $(G, -)$ has no $(K_{k+1}, -)$-minor, then $(G, -) \to (K_k, -)$.*

2.2 Packing number of signed graphs

Given a signed graph (G, σ), the *signature packing number*, or simply the packing number of (G, σ), denoted $\rho(G, \sigma)$, is the maximum number of signatures $\sigma_1, \sigma_2, \dots, \sigma_l$ such that each σ_i is switching equivalent to σ and the sets E_i^- are pairwise disjoint. If (G, σ) is equivalent to $(G, +)$, then by taking the all positive signature any arbitrary number of times, the conditions are satisfied. Hence, in this case we may set $\rho(G, +) = \infty$. Noting that this is a characterization of signed graphs with no negative cycle. For any other signed graph the packing number is a finite integer, which, moreover, admits the following basic upper bound.

Lemma 2.2. *Given a signed graph (G, σ), we have $\rho(G, \sigma) \leqslant g_-(G, \sigma)$.*

Proof. When (G, σ) has no negative cycle, then, by Theorem 2.1, it is equivalent to $(G, +)$ and in this case both $\rho(G, \sigma)$ and $g_-(G, \sigma)$ are set to be ∞. Otherwise, let C be a negative cycle of length $g_-(G, \sigma)$ and let $\sigma_1, \sigma_2, \dots, \sigma_l$ be a packing of (G, σ). Then each σ_i must assign a negative sign to at least one distinct edge of C, thus proving that l cannot be more than the length of C. □

The upper bound of this lemma in general can be far from equal. Indeed soon we will see how to find examples of signed graphs whose girth is as large as one wishes, but its packing number is 1. Furthermore, we will also observe that to decide if the equality holds in Lemma 2.2 for a general signed graph (G, σ) is an NP-complete problem. However, the study of sufficient conditions under which the equality in Lemma 2.2 holds captures a number of well studied theories in graph theory, with the 4-coloring problem and the Four-Color Theorem being among the most famous ones.

Given a signed graph (G, σ), we say it *packs* if $\rho(G, \sigma) = g_-(G, \sigma)$. Perhaps the most important signed graph that packs is $(K_4, -)$. In Figure 2.1 a 3-packing of $(K_4, -)$ is presented with indication of the switching that has resulted in each of the given signed graph. Observe that the negative edges of σ_1, σ_2 and σ_3 correspond to the (unique) proper 3-edge-coloring of K_4. This leads to further developments discussed about packing in this thesis.

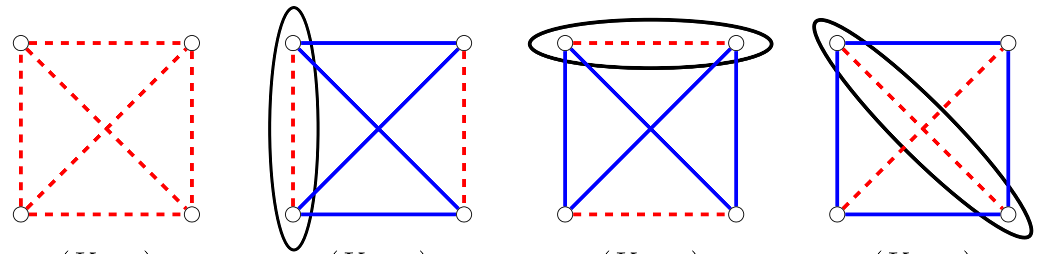

Figure 2.1: A 3-packing of $(K_4, -)$

On the other hand, as the smallest and perhaps simplest example of a simple signed graph whose packing number is 1 we have $(K_5, -)$. It can be easily checked that $\rho(K_5, -) = 1$ and thus $\rho(K_k, -) = 1$ for every $k \geqslant 5$. A strong conjecture, (see Chapter 3, Subsection 3.4.2 for a precise statement) is that under certain restriction this signed graph is also a minimal signed graph with respect to taking minor and having packing number 1.

The next lemmas are among earliest observation in the study of packing number of signed graphs.

Lemma 2.3. *Given a graph G which is not bipartite, the packing number of the signed graph $(G, -)$ is an odd number.*

Proof. Since G is not bipartite it has an odd cycle which is a negative cycle in $(G, -)$. Thus $\rho(G, -)$ is a finite number. Let $\sigma_1, \sigma_2, \dots, \sigma_{2l}$ be a packing of even order, that is to say the sets E_i^- are pairwise disjoint. Let $E_{2l+1} = E(G) - E_1^- \cup E_2^- \cdots \cup E_{2l}^-$. Then it is straightforward to verify that each odd cycle of G intersects E_{2l+1} in an odd number of edges and each even cycle intersects it in an even number of edges. Thus, by Theorem 2.1, the assignment σ_{2l+1} which assigns a negative sign to the edges in E_{2l+1} and positive sign to all the other edges produces a signed graph (G, σ_{2l+1}) equivalent to $(G, -)$. Therefore, the packing number of $(G, -)$ can never be an even number. $\square$

The proof of the next lemma is quite similar to the proof of the previous lemma and we skip it.

Lemma 2.4. *The packing number of any signed bipartite graph is an even number.*

The notion of packing signatures of a signed graph is developed from a discussion between R. Naserasr and T. Zaslavsky. A parallel and somewhat similar study is then carried on by a N. Lacasse, a Ph.D. student of Zaslavsky. His results are presented in [30] where the notion of *negating set* is employed to refer to the set of negative edges in a signed graph. In recent discussion with D. Cornaz we have learned that an equivalent form of the notion is mentioned [20]. This formulation together with the main contribution of [20] to this subject is mentioned in following subsections and Chapter 3.

2.2.1 Packing negative cycle covers

We point out here that the signature packing number is the same as negative cycle cover packing number of a signed graph. Based on this equivalence we have the following theorem of Gan and Johnson which can be regarded as the first result on this subject.

Given a signed graph (G, σ), a set CC of the edges of G is said to be *negative cycle cover* of (G, σ), or simply a cycle cover of (G, σ) if it contains at least one edge from each negative cycle of (G, σ). A collection $CC_1, CC_2, \ldots, CC_i$ of cycle covers of (G, σ) is said to be a *cycle cover packing* if no pair of them have a common element. The maximum number of cycle covers in a cycle cover packing is said to be *cycle cover packing number* of (G, σ). It turns out that the cycle cover packing number of any signed graph is equal to the signature packing number of it. This claim is immediately followed by employing a notion of minimality and a correspondence between minimal elements of the two notions.

Given a signed graph (G, σ), a signature σ' obtained from a switching of σ is said to be *minimal* if for no other switching σ'' we have $E^-_{\sigma''} \subseteq E^-_{\sigma'}$. Similarly, a cycle cover CC of (G, σ) is said to be minimal if no proper subset of it is a negative cycle cover of (G, σ). It is immediate that every equivalent signature of (G, σ) contains a minimal signature and that every cycle cover of it contains a minimal cycle cover. Thus, in each of the definition of the packing numbers if we restrict ourselves to the minimal elements of the corresponding set, we have the same result. That the signature packing number and the negative cycle packing number of a signed graph (G, σ) are equal then follows from the following lemma first proved in [26] (see Theorem 7 of this reference).

Lemma 2.5. *Given a signed graph (G, σ), every minimal cycle cover is a minimal signature and vice versa: every minimal signature is a minimal cycle cover.*

Restated in our language of packing signatures, one of the results of [20] is to show that K_3^2, that is the signed graph of Figure 2.2, is a minor minimal signed graph which does not pack. It is easily observed that $\rho(K_3^2) = 1$ while $g_-(K_3^2) = 2$. On the other hand:

Theorem 2.3. *[20] If a signed graph (G, σ) has no K_3^2-minor, then it packs, i.e., $\rho(G, \sigma) = g_-(G, \sigma)$.*

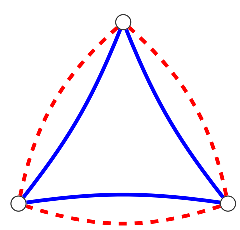

Figure 2.2: $\rho(K_3^2) = 1$

This result would also follow from the structural result of Gerarad's from Chapter 3 of [22], where he provide a decomposition theorem for the class of signed graphs with no K_3^2-minor.

Part II

Packing signatures in signed graphs

Chapter 3

Packing signed bipartite planar graphs

This chapter is based on the following paper:

[41] R. Naserasr and W. Yu. Packing signatures in signed graphs. Accepted for publication in SIAM J. Discrete Math., 2023.

The question of determining the packing number, introduced in Chapter 2, in a class of signed graphs captures or relates to some of the most prominent studies in graph theory. For example the four-color theorem can be restated as: For every planar simple graph G we have $\rho(G,-) \geqslant 3$. Motivated by Seymour's edge coloring conjecture and its relation with homomorphism to signed projective cubes, in this chapter, we consider the packing number of signed bipartite graphs.

In Section 3.1, we first connect the notion of packing number to the theory of homomorphism of signed graphs. Precisely, we show that for a signed graph (G,σ), $\rho(G,\sigma) \geqslant d+1$ if and only if (G,σ) admits a homomorphism to $\mathcal{SPC}_d^o$, where $\mathcal{SPC}_d^o$ is obtained from $\mathcal{SPC}_d$ by adding a positive loop to every vertex. In Section 3.2, we consider the relation between 4-coloring and packing number. In Section 3.3, we discuss packing number of signed planar graphs and some conjectures which are generalization of the 4-color theorem. In Section 3.4, we prove that : If G is a K_5-minor-free bipartite simple graph, then for any signature σ we have $\rho(G,\sigma) \geqslant 4$. The statement is shown to be strictly stronger than the four-color theorem and is proved assuming it.

3.1 Signed Projective Cubes

The *signed projective cube* of dimension d, denoted $\mathcal{SPC}_d$, is a signed graph on $\mathbb{Z}_2^d$ as the vertex set where two vertices are adjacent by a positive edge if they are at hamming distance 1 and by a negative edge if they are at hamming distance d. That is to say $\mathcal{SPC}_d$ is built from the hypercube of dimension d by taking all the edges to be positive and adding a negative edge between each pair of antipodal vertices. For the sake of completeness we also define $\mathcal{SPC}_0$ to be the signed graph on one vertex with a negative loop. The first few signed projective cubes are depicted in Figure 3.1. For equivalent definitions of $\mathcal{SPC}_d$ and for a proof of the following lemma we refer to [37] and [39].

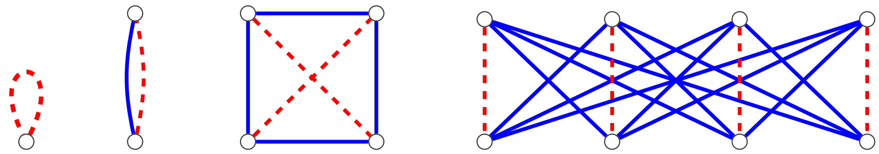

Figure 3.1: $\mathcal{SPC}_d$ for $d \in \{0, 1, 2, 3\}$

Lemma 3.1. *For odd values of d, $\mathcal{SPC}_d \in \mathcal{G}_{10}$ and for even values of d, $\mathcal{SPC}_d \in \mathcal{G}_{11}$. Moreover $g_-(\mathcal{SPC}_d) = d + 1$.*

Given the signed graph $\mathcal{SPC}_d$, one may label its positive edges by the coordinate that is the witness of the hamming distance 1 between its two ends and label negative edges by J. It is easily observed that this labeling is a proper edge-coloring of the underlying graph $\mathcal{PC}_d$. Furthermore, in this edge-coloring each pair of colors induces an edge cut of $\mathcal{PC}_d$. Thus the signed graph $(\mathcal{PC}_d, \pi_i)$, where π_i assigns a negative sign to the edges labeled i for $i \leqslant d$ and to the edges labeled J for $i = d + 1$, is switching equivalent to $\mathcal{SPC}_d$. As no pair of these $d + 1$ signatures share a common negative edge, and together with $g_-(\mathcal{SPC}_d) = d + 1$ we have:

Lemma 3.2. *Given a non negative integer d, the signed graph $\mathcal{SPC}_d$ packs. More precisely $\rho(\mathcal{SPC}_d) = g_-(\mathcal{SPC}_d) = d + 1$.*

Observe that in the above example of $(d + 1)$-packing of $\mathcal{SPC}_d$, we not only find examples of signatures without sharing a negative edge, but also partition the set of edges of $\mathcal{PC}_d$ into sets of negative edges of the signatures. It is shown in [37] that the problem of decomposing edges of a signed graph into $d + 1$ sets, each corresponding to the negative edges of an equivalent signature, is equivalent to a homomorphism problem where the signed graph $\mathcal{SPC}_d$ plays the role of universal target. More precisely, we have the following theorem:

Theorem 3.1. *[37] Given a non negative integer d, the edge set of a signed graph (G, σ) can be decomposed into $d+1$ sets $E_1, E_2, \ldots, E_{d+1}$, with each E_i being the set of negative edges of a switching equivalent signed graph (G, σ_i), if and only if $(G, \sigma) \to \mathcal{SPC}_d$.*

Here using a modification on a signed projective cube we introduce a variant of this theorem which captures packing problems of signed graphs where the edge set is not necessarily decomposed, but rather a number of disjoint subsets are selected.

Definition 3.1. We define $\mathcal{SPC}_d^o$ to be the signed graph obtained from the signed projective cube of dimension d by adding a positive loop to each of its vertices.

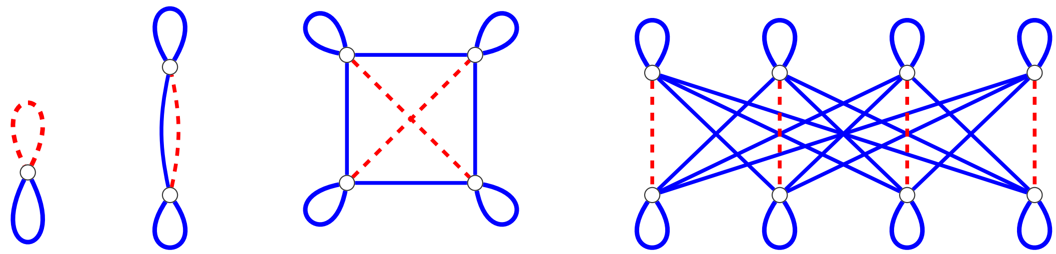

Figure 3.2: $\mathcal{SPC}_d^o$ for $d \in \{0, 1, 2, 3\}$

The first few examples of $\mathcal{SPC}_d^o$ are given in Figure 3.2.

Theorem 3.2. *Given a non negative integer d, for a signed graph (G, σ), we have $\rho(G, \sigma) \geqslant d + 1$ if and only if $(G, \sigma) \to \mathcal{SPC}_d^o$.*

Proof. Let (G, σ) be a signed graph. First suppose $(G, \sigma) \to \mathcal{SPC}_d^o$. Following the discussion on equivalent signatures of $\mathcal{SPC}_d$, we denote by $(\mathcal{SPC}_d^o, \pi_i)$ the signed graph on $\mathcal{SPC}_d^o$ where for $i = 1, 2, \ldots, d$, the edges labeled i are the negative edges and for $i = d + 1$ the edges labeled J are the negative edges. Then for each i, $i = 1, 2, \ldots, d + 1$, the set of edges of G mapped to the negative edges of $(\mathcal{SPC}_d^o, \pi_i)$ forms the set of negative edges of a signature σ_i of G which is equivalent to σ. As these sets are disjoint, we have $\rho(G, \sigma) \geqslant d + 1$.

For the inverse assume that $\rho(G, \sigma) \geqslant d + 1$. Thus there are at least $d + 1$ signatures $\sigma_1, \sigma_2, \ldots, \sigma_{d+1}$ such that each σ_i is switching equivalent to σ and the sets E_i^- are pairwise disjoint. Let $E' = E(G) - \bigcup\limits_{i=1}^{d+1} E_i^-$. Let G' be the graph obtained by contracting all edges in E'. Let σ_i' be the signature on G' induced by the signature σ_i on G; that is to say the set of edges assigned a negative sign by σ_i' is the set E_i^-.

We claim that each pair of σ_i' and σ_j' are switching equivalent signatures on G'. This follows from discussion in Section 4 of [51], and can verified directly as well. Since σ_i and σ_j are switching equivalent signatures on G, there is a cut $(X, V \backslash X)$ in G such that if (G, σ_i) is switched on X we get (G, σ_j). As an edge uv of E' is positive in both of these signatures, u and v should either be both in X or both in $V \backslash X$. Thus by contracting the edges in E' the edge cut $(X, V \backslash X)$ would induce an edge cut $(X', V(G') \backslash X')$ of G'. Starting with the signed graph (G', σ_i'), a switching on the edge cut $(X', V(G') \backslash X')$ would then result in (G', σ_j'), thus proving that σ_i' and σ_j' are switching equivalent.

Thus the edges of G' are decomposed into $d + 1$ disjoint parts as the negative edges of the signatures σ_i', and, therefore, by Theorem 3.1, (G', σ_1') admits a homomorphism to $\mathcal{SPC}_d$. This then easily extends to a homomorphism of (G, σ_1) to $\mathcal{SPC}_d^o$ by noting that the edges in E' are positive in (G, σ_1) and are mapped to the positive loops. $\square$

Following the proof technique of Lemmas 2.3 and 2.4 we have the following lemma which connects Theorem 3.1 and Theorem 3.2.

Lemma 3.3. *A signed graph (G, σ) belongs to $\mathcal{G}_{10} \cup \mathcal{G}_{11}$ if and only if its edge set can be partitioned into sets $E_1, E_2, \ldots, E_l$, for some integer l, each of which is the set of negative edges of a signature σ_i equivalent to σ.*

Theorem 3.3. *Given a signed graph (G, σ) of packing number $d+1$, we have $(G, \sigma) \in \mathcal{G}_{10} \cup \mathcal{G}_{11}$ if and only if $(G, \sigma) \to \mathcal{SPC}_d$.*

3.2 4-coloring of graphs and Packing signed graphs

Since $\mathcal{SPC}_2$ is switching equivalent to $(K_4, -)$, and considering the fact that for a non bipartite graph G the packing number of $(G, -)$ is always an odd number we have the following.

Theorem 3.4. *A graph G is 4-colorable if and only if $\rho(G, -) \geqslant 2$.*

Proof. If G is bipartite, then $\rho(G, -) = \infty$ and G is 4-colorable, in which case there is nothing left to prove. Thus we assume G is not bipartite.

A graph G is 4-colorable if and only if it admits a homomorphism to K_4. By Theorem 2.2, that is to say: A graph G is 4-colorable if and only if the signed graph $(G, -)$ admits a homomorphism to $(K_4, -)$. Since $(K_4, -)$ is switching equivalent to $\mathcal{SPC}_2$, we have: a graph G is 4-colorable if and only if $(G, -)$ maps to $\mathcal{SPC}_2$. As $(G, -) \in \mathcal{G}_{11}$, by Theorem 3.3 and Theorem 3.1, G is 4-colorable if and only it has packing number at least 3. Finally, since $(G, -) \in \mathcal{G}_{11}$, and by Lemma 2.3, a graph G is 4-colorable if and only if $(G, -)$ has a packing of order 2. □

Using the four-color theorem, or rather a strengthening of it on the class of K_5-minor-free graphs, and by Lemma 2.3, we have the following corollary.

Corollary 3.1. *Given a K_5-minor-free graph G with no loop, we have $\rho(G, -) \geqslant 3$.*

Given a graph G, a signed bipartite graph $S(G)$ is defined as follows: vertices of $S(G)$ consist of vertices of G as one part of $S(G)$ and for each edge uv two vertices labeled x_{uv}, y_{uv} on the other part of $S(G)$. For each edge uv of G then we build a 4-cycle $ux_{uv}vy_{uv}$. The signature of $S(G)$ is an assignment π_0 which assigns a negative sign to exactly one edge of each 4-cycle of the constructed bipartite graph. We note that the choice of π_0 is arbitrary and that different choices are not necessarily switching equivalent but they result in (switching) isomorphic graphs. This construction was first introduced in [38]. The following theorem is implied using a result of [38] and Theorem 3.3.

Theorem 3.5. *Given a simple graph G, we have $\rho(G, -) \geqslant 3$ if and only if $\rho(S(G)) \geqslant 4$ (for any choice of π_0).*

Thus to prove that $\rho(S(G)) \geqslant 4$ is the same as proving that G is four-colorable. Noting that for every planar graph G, the associated signed graph $S(G)$ is a signed bipartite planar graph, to claim that every signed planar simple bipartite graph has packing number at least 4 is stronger than the four-color theorem. This is proved to be the case and is discussed in more details in the next section.

3.3 Packing signed planar graphs

An example of a signed planar simple graph which does not map to $\mathcal{SPC}_1^o$ is given in [36]. Combined with Theorem 3.2 we have the following.

Proposition 3.1. *There exists a signed planar simple graph (G, σ) satisfying $\rho(G, \sigma) = 1$.*

Thus in Corollary 3.1, the assumption on the signature, i.e., that $(G, \sigma) \in \mathcal{G}_{11}$, is essential. However, with this kind of restriction a generalization of the 4CT can be proposed as follows.

Conjecture 3.1. *Every signed planar graph in $\mathcal{G}_{11} \cup \mathcal{G}_{10}$ packs.*

That is to say: given a signed planar graph $(G, \sigma) \in \mathcal{G}_{11} \cup \mathcal{G}_{10}$, the packing number of (G, σ) is equal to the negative girth of (G, σ). We note that a signed connected graph is in $\mathcal{G}_{11} \cup \mathcal{G}_{10}$ if it has no positive odd closed walk, i.e. $g_{01}(G, \sigma) = \infty$.

From the discussion of Section 3.1 it follows that Conjecture 3.1 is equivalent to:

Conjecture 3.2. *Given a signed planar graph in $\mathcal{G}_{11} \cup \mathcal{G}_{10}$, if $g_{-}(G, \sigma) = d + 1$, then $(G, \sigma) \to \mathcal{SPC}_d$.*

This conjecture, which is partly proposed in [34] and partly in [24], is shown [34] and [37] to be equivalent to the following conjecture, which is a restricted version of P. Seymour.

Conjecture 3.3. *Given a k-regular planar graph, it is k-edge-colorable if for each set X of odd number of vertices the edge cut $(X, V \backslash X)$ is of size at least k.*

It is easily observed that the connectivity condition in this conjecture is necessary. The conjecture is a generalization of Tait's reformulation of the 4CT. Thus the case $k = 3$ is implied by the 4CT. The cases $k = 4, 5$ were settled by B. Guenin, in 2002 using the notion of packing T-joins but it remains unpublished. The claimed proof is based on induction on k, thus the 4CT (the case $k = 3$) is assumed. The result is extended by several authors for $k = 6, 7, 8$. Our result in this work, based on the notion of packing, implies a proof of the case $k = 4$. Our proof has some similar elements to that of Guenin. There are advantages in our approach, a notable one being that: since faces are not needed, our result works for any minor closed family of 4-colorable graphs. The largest of those is the class of K_5-minor-free graphs, but taking some smaller class one may get a proof without using the 4CT. More precisely we prove that:

Theorem 3.6. *Any signed bipartite simple K_5-minor-free graph has a packing number at least 4.*

To prove Theorem 3.6 we establish a number of lemmas that could be of use for the general case of Conjecture 3.1. These are collected in the next section.

3.4 Packing and minors

The advantage of Conjecture 3.1 is that induction on the negative girth looks possible and indeed we will prove the case of negative girth being 4 use negative girth 3 (which is equivalent to the 4CT). This is based on the following easy lemma. We recall that for a subset E_1 of the edges of a graph G, the graph obtained from contracting all edges in G is denoted by G/E_1.

Lemma 3.4. *Let (G, σ_1) and (G, σ'_1) be two switching equivalent signed graphs with no common negative edge. Then $\rho(G, \sigma_1) \geqslant \rho(G/E_1, E'_1) + 1$, where E_1 and E'_1 are the sets of the negative edges of (G, σ_1) and (G, σ'_1), respectively.*

Proof. Let $\sigma_2, \sigma_3, \ldots, \sigma_{k+1}$ be k signatures on G/E_1 such that each is equivalent to $(G/E_1, E'_1)$ and that no pair of them have a common negative edge. Let E_2, E_3, $\ldots$, E_{k+1} be the set of negative edges in $(G/E_1, \sigma_2)$, $(G/E_1, \sigma_3)$, $\ldots$, $(G/E_1, \sigma_{k+1})$, respectively. Then it is quite straightforward to check that (G, E_1), (G, E_2), (G, E_3), $\ldots$, (G, E_{k+1}) is a packing of (G, σ_1). $\square$

In applying this lemma one should note that if (G, σ_1) is in $\mathcal{G}_{11}$, then $(G/E_1, E'_1)$ is in $\mathcal{G}_{10}$ and that conversely, if $(G, \sigma_1) \in \mathcal{G}_{10}$, then $(G/E_1, E'_1) \in \mathcal{G}_{11}$. Thus if we are attempting to prove that for a minor closed family $\mathcal{C}$ of graphs, every signed graph (G, σ), $(G, \sigma) \in \mathcal{G}_{11} \cup \mathcal{G}_{10}$ and $G \in \mathcal{C}$, packs, then in an approach which is based on induction on the negative girth of (G, σ), assuming the claim holds as long as $g_-(G, \sigma) \leqslant k$, and given a signed graph (G, σ) in the class satisfying $g_-(G, \sigma) = k + 1$, it would be enough to find signatures σ_1 and σ'_1, each equivalent to σ and such that $g_-(G/E_1, \sigma'_1) \geqslant k$.

When (G, σ) is in $\mathcal{G}_{10}$, finding σ'_1 or rather E'_1 is quite simple, it would be enough to set $E'_1 : E \backslash E_1$. Thus in this case the main task in hand would be to find an appropriate σ_1. When (G, σ) is in $\mathcal{G}_{11}$, then we must provide both σ_1 and σ'_1 when applying this technique. However, in this case finding σ' can also be done with a condition on σ_1: let (G, σ_1) be a switching of $(G, -)$ with the property that every negative cycle of $(G, -)$, that is every odd cycle of G, has at least one (therefore, at least 2) positive edges. Thus in the minor (G/E_1) of G every negative closed walk of G has an image which is a nontrivial closed walk of G/E_1. The set of all these closed walks have a θ-property: that if we take three $x - y$ walks P_1, P_2 and P_3, then of the three closed walks P_1P_2, P_1P_3 and P_2P_3 either none or exactly two of them are in the set. Then it follows from Theorem 10 of [39] that this set of closed walks is the set of negative closed walks of a signature on G/E_1. Taking E'_1 as the set of negative edges of such a signature then works.

Thus based on this discussion, Conjecture 3.1 is equivalent to the following conjecture:

Conjecture 3.4. *Given a signed planar graph $(G, \sigma) \in \mathcal{G}_{11} \cup \mathcal{G}_{10}$, there is an equivalent signature σ_1 such that every negative cycle of (G, σ) has at least $g_-(G, \sigma)) - 1$ positive edges.*

Theorem 3.7. *Conjecture 3.1 and Conjecture 3.4 are equivalent.*

Proof. That Conjecture 3.1 implies Conjecture 3.4 is straightforward: if $\sigma_1, \sigma_2, \ldots, \sigma_k$ is a packing of (G, σ), then any of σ_i's satisfies the condition of Conjecture 3.4: every negative cycle of (G, σ_i) has at least one negative edge in each of (G, σ_j), $j \neq i$, all of which are positive in (G, σ_i).

Suppose Conjecture 3.4 holds. Let (G, σ) be a counterexample to Conjecture 3.1 of minimum possible negative girth, say k. By the statement of Conjecture 3.4 there is a switching equivalent signature σ_1 where each negative cycle has at least $k-1$ positive edges. Considering the signed graph $(G/E_1, \sigma_1')$, where σ_1' is a signature equivalent to σ but disjoint from it, the negative girth is $k-1$. By our choice of (G, σ), which has minimal negative girth among all counterexamples, $(G/E_1, \sigma_1')$ packs. Thus there are signatures $\sigma_2, \sigma_3, \ldots \sigma_k$ where no pair of them have a common negative edge. Together with E_1, then they correspond to signatures $\sigma_1, \sigma_2, \ldots, \sigma_k$ proving that (G, σ) packs. □

Following this formulation, given a signed graph (G, σ) of negative girth k, a negative cycle whose number of positive edges is (strictly) less than $k-1$ will be referred to as *super negative* cycle. Thus Conjecture 3.4, and, therefore, Conjecture 3.1, are to say that any planar signed graph $(G, \sigma) \in \mathcal{G}_{11} \cup \mathcal{G}_{10}$ can be switched so that it has no super negative cycle.

There are a couple of important remarks to make here: first is that we did not really use the assumption of planarity here, rather we used the fact that we are working with a minor closed family of graphs, or even more precisely, we want the minor G/E_1 to be in our family. The second remark is that if we restrict both conjectures on subclass of signed graphs of negative girth at most k, then these restricted versions are still equivalent.

Following these observation, we would like to work with a minor closed family $\mathcal{C}$ of graphs such that any signed graph (G, σ) with $G \in \mathcal{C}$ and $(G, \sigma) \in \mathcal{G}_{11} \cup \mathcal{G}_{10}$ packs. If we take all signed graphs $(G, -)$ in this family, where G is a simple graph, then the fact that $(G, -)$ packs implies, in particular, that G is 4-colorable. Thus, in particular, K_5 is not in $\mathcal{C}$ and as $\mathcal{C}$ is a minor closed family, we are working with a subclass of K_5-minor-free graphs. One may assume that $\mathcal{C}$ is indeed the class of K_5-minor-free graphs, but there is advantage in this general statement which will be pointed out in Subsection 3.4.2.

Before continuing, we state a couple of facts on K_5-minor-free graphs.

The first is the following classic theorem of Wagner on characterization of K_5-minor-free graph. Here W is the graph of Figure 3.3.

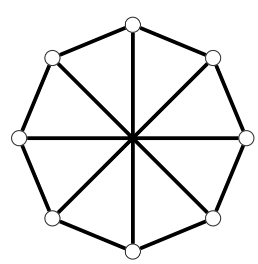

Figure 3.3: Wagner graph

Theorem 3.8. *(Wagner) Every edge-maximal graph with no K_5-minor can be obtained by means of 3-sum and 2-sum, starting from planar triangulations and copies of W.*

A 3-sum of two graphs G and H is to identify the vertices of one triangle of G with the vertices of a triangle of H. Similarly, their 2-sum is to identify the vertices of an edge from G with the vertices of an edge from H. A first and classic corollary of this decomposition theorem is that, the four-color theorem can be extended to the class of K_5-minor-free graphs, this is a classic application of this decomposition theorem. A second corollary is to extend the application of the Euler formula to bound the number of edges of a triangle-free members of the class, we give a proof of this folklore fact for the sake of completeness.

Proposition 3.2. *If G is a K_5-minor-free graph of girth at least 4, then* $|E(G)| \leqslant 2|V(G)| - 4$.

Proof. First we build a graph G' from G by adding edges to make it edge-maximal while it remains K_5-minor-free. Obviously, G is a spanning subgraph of G'. Then by Theorem 3.8, G' is obtained from 3-sum or 2-sum of planar triangulations and copies of W. Suppose G' is obtained by clique-sums of $G'_1, G'_2, \ldots, G'_n$. Without loss of generality, let G''_i be the clique-sums of $G'_1, \ldots, G'_i$. Let G_i be the subgraph of G''_i contained in G, let H_i be the subgraph of G'_i contained in G. Then $G = G_n$, and it suffices to prove that $|E(G_n)| \leqslant 2|V(G_n)| - 4$.

We first claim that the inequality holds for each H_i. That is because each H_i is either planar and triangle-free, in which case $|E(G_1)| \leqslant 2|V(G_1)| - 4$ by application of the Euler formula, or it is a spanning subgraph of W, and the inequality holds for W itself. Thus in particular $G_1 = H_1$ satisfies the conditions. We complete the proof by induction on i, showing that each G_i satisfies the bound. That is because if G''_i is obtained from 3-sum of G''_{i-1} and G'_i, then G_i is formed from G_{i-1} and H_i by identifying three vertices and at most two edges. Since they both satisfy the inequality, G_i also satisfies it. If G''_i is obtained from 2-sum of G''_{i-1} and G'_i, then G_i is formed from G_{i-1} and H_i by identifying two vertices and at most one edge, and similarly, G_i also satisfies the inequality. □

We are now ready to state and prove the following.

Theorem 3.9. *Let $\mathcal{C}$ be a minor closed family of graphs whose members are 4-colorable. Then for any bipartite simple graph G in $\mathcal{C}$ and for any signature σ we have $\rho(G, \sigma) \geqslant 4$.*

Proof. Assume that (G, σ) is a minimal counterexample to the theorem. That is to say that G is a simple bipartite graph in $\mathcal{C}$ with a signature σ such that $\rho(G, \sigma) = 2$ and that for any edge e of G, the signed bipartite graph $(G - e, \sigma)$ has packing number at least 4.

Here the signature in $(G - e, \sigma)$ is the restriction of the signature of (G, σ), thus, with a minor abuse of notation, we use σ to denote both. Furthermore, if $(G - e, \sigma')$ is obtained from $(G - e, \sigma)$ by switching at a subset X of vertices, then we may use (G, σ')

to denote the signature which is obtained from (G,σ) by switching at the same vertex set X, in this case $(G-e,\sigma')$ will be induced signed subgraph of (G,σ').

With this notation and with the assumption on the minimality of (G,σ), we conclude that for each edge e of G, there are four signatures σ_1, σ_2, σ_3 and σ_4 such that any pair of them either have no common negative edge, or e is their only common negative edge. We recall, by Theorem 3.7, (G,σ) is also a minimal counterexample to Conjecture 3.4. As we are considering negative girth to be 4, given a signature, a super negative cycle is a negative cycle with only one positive edge. If for any signature equivalent to σ, in particular for one of the signatures σ_i, $i=1,2,3,4$, the signed graph (G,σ_i) has no super negative cycle then we are done. On the other hand $(G-e,\sigma_i)$ has no super negative cycle for $i=1,2,3,4$ because each negative cycle has at least one negative edge in each σ_i which is a positive edge in the other three signatures. Thus each (G,σ_i), $i=1,2,3,4$, must have a super negative cycle which contains e.

One easily observes that replacing a signature σ_i, $i=1,2,3,4$, with a minimal signature contained in σ_i may only decrease the number of super negative cycles. Thus we may assume each σ_i is a minimal signature. This in particular implies that:

$$\text{not all edges incident to the same vertex are negative in a given } \sigma_i. \qquad (3.1)$$

Let $e=uv$ be an edge where $d(u)=2$. Let σ_1, σ_2, σ_3, σ_4 be a 4-packing of $(G-e,\sigma)$ consisting of four minimal signatures. We claim that, for each signature σ_i, $i=1,2,3,4$, at least one super negative cycle C_i in (G,σ_i) has the following property:

P1. Except possibly the two vertices of the only positive edge of C_i, every (other) vertex of C_i has a degree at least 4 in G.

Since σ_i's are assumed to be minimal, and by (3.1), in none of (G,σ_i) the two edges incident to u are negative. They cannot be both positive either, as otherwise (G,σ_i) has no super negative cycle and we are done. If necessary by switching at u we may assume $e=uv$ is the negative edge in each of (G,σ_i) and that the other edge incident to u, say uw, is positive in all of them. We now consider a super negative cycle C_i of (G,σ_i). Observe that, as this cycle must contain e, and since u is a vertex of degree 2, it must also contain uw, and thus uw is its only positive edge. Let x be a vertex on C_i which is of degree 2 or 3 in G and $x\notin\{u,w\}$. Then x is not of degree 2 because of (3.1), thus $d(x)=3$. Let xy be the edge incident to x which is not on C_i. Observe that, again by (3.1), xy is a positive edge of (G,σ_i). Moreover, as $x\notin\{u,w\}$, xy is distinct from uw. Thus no super negative cycle of (G,σ_i) contains the edge xy. Let σ_i' be the signature on G obtained from a switching at the vertex x. Observe the following 3 facts: 1. C_i is not a super negative cycle in (G,σ_i'), 2. Because $x\notin\{u,w\}$, the number of positive edges incident to u is not decreased but it may have gone up if $x=v$. 3. If C_i' is a super negative cycle of (G,σ'), then C_i' is also a super negative cycle of (G,σ_i) and moreover, signs of each edge of C_i' are the same in both (G,σ_i) and (G,σ_i'). Thus if a super negative cycle of (G,σ_i') satisfies the conditions of **P**1 then we are done, otherwise we repeat the process. As we are working with a finite graph, and the number of super negative cycles is finite, at the end either we find a super negative cycle that satisfies the conditions of **P**1, or we obtain a signature with no super negative cycle in which case we can find a packing of four signatures and we are done.

In conclusion, we have a 4-packing σ_1, σ_2, σ_3, σ_4 of $(G-e,\sigma)$ with the property that each (G,σ_i), $i=1,2,3,4$, contains a super negative cycle C_i in which uw is the only positive edge, and, except for u and w, every other vertex on C_i is of degree at least 4 in G. Let x_i be the neighbour of v on C_i distinct from u. Observe that as G is bipartite, x_i is also distinct from w. We observe, furthermore, that any pair of the signatures σ_i and σ_j, $i,j \in \{1,2,3,4\}$, can only have $e=uv$ as the common negative edge. We conclude that v has (at least) four neighbours each of which is of degree 4.

This argument can be repeated exchanging the roles of v and w, thus we conclude that:

Claim 3.1. *For each vertex u of degree 2, each of its neighbors v and w has four neighbors each of degree at least 4.*

Next we aim to prove a similar claim for the neighborhood of a 3-vertex. Proofs are quite similar, but we need to take care of further details.

Let u be a vertex of degree 3 and let v, w and t be its three neighbors. Consider $e=uv$ and let σ_1, σ_2, σ_3, σ_4 be a 4-packing of $(G-e,\sigma)$ consisting of four minimal signatures. We first observe that in each of (G,σ_i), $i=1,2,3,4$, not all three edges uv, uw, ut are of a same sign. That is because three of them being negative would contradict our choice of σ_i's being minimal and three of them being positive will leave no room for a super negative cycle in (G,σ_i) containing uv, noting that there is also no super negative cycle in $(G-e,\sigma_i)$ by our choice of σ_1, σ_2, σ_3, σ_4. If for any of σ_i the signed graph (G,σ_i) contains two negative edges incident to u, then we will switch at u to get a signature σ'_i.

So altogether we will work with signatures σ'_1, σ'_2, σ'_3, σ'_4 such that in each signed graph (G,σ'_i), the signature σ'_i assigns one negative and two positive signs to the edges uv, uw, ut and σ'_i is either the same as σ_i, or is obtained from σ_i by switching at u. Observe that, by the choice of σ_i, $i=1,2,3,4$, any pair of signatures among σ'_i's have at most one common negative edge, and if so, that edge is one of uv, uw, ut. We may further modify σ'_i's to have them as minimal signatures. One may remind the reader again that replacing a σ'_i with potentially minimal subset would not create a new intersection among σ'_i's and that the only affect such a replacement may have on super negative cycles is to kill off some.

We claim again that, for each signature σ'_i, $i=1,2,3,4$, at least one super negative cycle C_i in (G,σ'_i) has the following property: every vertex of C_i not incident with the positive edge of C_i has degree at least 4 in G.

To prove the claim we first note that C_i is also a super negative cycle of (G,σ_i). That is because first of all edges not incident to u that are negative in (G,σ'_i) are also negative in (G,σ_i). Secondly, since the only positive edge of C_i is incident to u, each edge of C_i which is not incident to u is negative in both (G,σ'_i) and, therefore, in (G,σ_i). Thirdly, since C_i is a negative cycle of (G,σ), and as G is bipartite, in both (G,σ_i) and (G,σ'_i) one of the two edges incident with u is positive and the other is negative.

We conclude two facts from this: 1. that every super negative cycle of (G, σ_i') must contain the edge uv, and, therefore, 2. the positive edge of every super negative cycle of (G, σ_i') is incident to u. We note that this is not necessarily true for (G, σ_i).

We now consider a shortest super negative cycle C_i of (G, σ_i') and assume that it contains a vertex x not incident to the positive edge of C_i and that $d(x) \leqslant 3$. Once again by the fact that σ_i' is a minimal signature, we conclude that x must be of degree exactly 3 and that the edge xy which is the edge incident with x but not in C_i must be positive. We claim that $y \neq u$. Otherwise, since C_i must contain u as well, xy is a chord of C_i. Then xy creates two cycles with C_i and the part that does not contain the positive edge of C_i is a super negative cycle of (G, σ_i') but it is shorter than C_i, contradicting the choice of C_i.

That $y \neq u$ implies that no super negative cycle of (G, σ_i') contains xy. Let σ_i'' be the signature obtained from a switching of (G, σ_i') at x. What we have observed is that: 1. C_i, which was a super negative cycle of (G, σ_i'), is not a super negative cycle in (G, σ_i''), and 2. for every super negative cycle C of (G, σ_i'') each edge of C has the same sign in (G, σ_i'') and (G, σ_i'). We observe that σ_i'' is not necessarily minimal, however, replacing it with a minimal signature can only kill off some super negative cycles without any change on the signs of edges of the remaining one. Thus the remaining super negative cycles are the super negative cycles of (G, σ_i') without any change to the signs of their edges. We continue this process, if we end up with a signature where there is no super negative cycles, then we have found a 4-packing of (G, σ). Else we must end up with a super negative cycle C_i' where each vertex not incident with the positive edge of C_i' is of degree at least 4 in G. Since we have retained the sign of super negative cycles during the process, C_i' is also super negative cycle of (G, σ_i') with the property that each vertex not incident with the positive edge of C_i' is of degree at least 4 in G. We recall that each super negative cycle of (G, σ_i') must contain the edge $e = uv$ and that its only positive edge must be incident to u. Thus if vz_i, $z_i \neq u$, is an edge of C_i', then z_i is of degree at least 4 in G. As this must be true for every i, $i = 1, 2, 3, 4$, we have proved the following claim.

Claim 3.2. *If v is a vertex of degree 3 in G, then its nieghbors x, y and z each has at least four neighbors of degree at least 4.*

We may now employ the discharging technique to obtain a contradiction.

Discharging procedure

The initial charge of each vertex v is defined as: $\omega(v) = d(v)$. As G is K_5-minor-free and bipartite (thus triangle-free), by Proposition 3.2, we have $\sum\limits_{v \in V(G)} \omega(v) \leqslant 2|V(G)| - 8$. However, the following discharging rule will redistribute charges such that each vertex has a charge of at least 4, contradicting this formula.

(R1) Each vertex of degree 2 or 3 receives a charge of 1 from each of its neighbors.

Our two claims imply that for vertex v of degree 2 or 3 all neighbors are of degree at least 5, and thus while v gets a charge of 1 from each of its neighbors, it looses no charges, and thus has a final charge of at least 4. On the other hand a neighbor of such a vertex v has at least four vertices each of which is of degree at least 4, thus its charge will never go below 4. □

Corollary 3.2. *Any signed bipartite K_5-minor-free graph admits a homomorphism to SPC_3.*

3.4.1 Algorithmic conclusion

We recall that the proof of the four-color theorem provided in [45] leads to a quadratic time algorithm for 4-coloring of planar graphs. More precisely, that is an algorithm A which takes as an input a simple planar graph G and gives as an output a proper 4-coloring of G, in time $O(|V(G)|^2)$. Using the Wagner decomposition theorem, this works on the class of K_5-minor-free graphs as well. This is equivalent to giving a 3-packing of the signed graph $(G, -)$ as we discussed before. We may then use this algorithm to give an algorithm BP which takes as an input a signed bipartite planar simple graph (G, σ) and gives, as an output, a 4-packing of (G, σ) in time $O(|V(G)|^3)$. This follows easily from our proof: Since G is planar, bipartite and simple, it has at most $2n-4$ edges. We may simply assume G is 2-connected as one may combine solutions on distinct 2-connected blocks. Our discharging proof implies that G has either a vertex v of degree 2 where at least one of the neighbors, say x, has at most 3 neighbors of degree at least 4, or it has a vertex u of degree 3 each of whose neighbors have at most 3 neighbors of degree 4 or more.

Having found such a vertex v or u, that can be done in a linear time, we remove from (G, σ) an edge e incident to v or u. Assume a solution $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ is provided for $(G - e, \sigma)$. By the proof given in the previous section we know one of the four signed graphs $(G, \sigma_1), (G, \sigma_2), (G, \sigma_3), (G, \sigma_4)$ has no super negative cycle. This can be verified by checking for a loop in the graphs G/E_1, G/E_2, G/E_3 and G/E_4, noting that contracting these edges and looking for a loop can all be done in linear time. Suppose G/E_4 has no loop. Then we apply algorithm A on the graph G/E_4 to get signatures σ'_1, σ'_2, σ'_3. These three signatures together with σ_4 form a 4-packing of (G, σ).

To find a solution for $(G - e, \sigma)$, which we had assumed in the argument above, one may repeat the same process. Assuming G is on n vertices, since G has at most $2n - 8$ edges, the algorithm A might be recalled at most $2n - 8$ times. As algorithm A runs in time $O(n^2)$, the running time of the full algorithm is $O(n^3)$.

Mapping a signed bipartite graph (G, σ) to SPC_3, given a 3-packing $\sigma_1, \sigma_2, \sigma_3$, can be done in linear time: label negative edge in (G, σ_1) by 001, those in (G, σ_2) by 010, ones in (G, σ_3) by 100 and then label the remaining edges 111 noting that they form the negative edges of an equivalent signature. Observe that sum of the labels of the edges in each cycle is 000. Now for each connected component of (G, σ) take an arbitrary vertex, say x and map it to the vertex 000 of SPC_3. Then for a vertex y in the same component as x, take an xy path P and map y to sum of labels of edges of the path P. It can be readily verified that this is a mapping of (G, σ) to SPC_3.

We note that the algorithm works the same for signed bipartite K_5-minor-free graphs. However, the planar case has the following application on the dual.

Corollary 3.3. *Given a 4-regular planar multigraph G where each set X of odd number of vertices is connected to $V\backslash X$ by at least 4 edges, we have $\chi'(G) = 4$. Moreover, a 4-edge-coloring can be found in time $O(|F(G)|^3)$, where $|F(G)|$ is the number of faces of G.*

3.4.2 Concluding remarks

We introduced the notion of packing signatures in signed graph and we established connections with a number of problems such as 4-coloring of graphs, edge-coloring of planar graphs, etc.

We proved that given a minor closed family $\mathcal{C}$ of 4-colorable graphs, for any bipartite simple graph in $\mathcal{C}$ and any signature σ on it, the packing number of (G, σ) is at least 4. The largest family to which this result may apply is the class of K_5-minor-free graphs where 4-colorability of a general member is established by the four-color theorem. However, if we take smaller classes where 4-coloring can be verified without the use of the four-color theorem, then the result on the packing number will also be independent of the four-color theorem. An interesting case to mention is the following.

Theorem 3.10. *Given a signed bipartite simple graph (G, σ) where G has treewidth at most 3, we have $\rho(G, \sigma) \geqslant 4$.*

Corollary 3.4. *Every signed bipartite simple graph of treewidth at most 3 admits a homomorphism to SPC_3.*

The class of graphs of treewidth at most 3 is a minor closed family of graphs that is a subclass of K_5-minor-free graphs. More precisely, as proved in [1], it consists of graphs which do not have any of the four graphs of Figure 3.4 as a minor. That loop-free members of this class are 4-colorable follows from the fact that edge-maximal elements are 3-trees. Thus Theorem 3.10 is proved without using the four-color theorem.

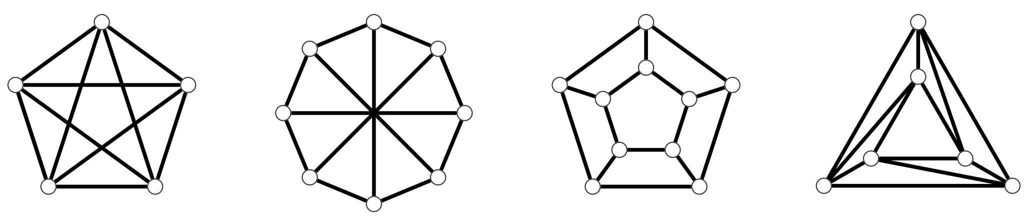

Figure 3.4: Forbidden minors for graphs of treewidth at most 3

On the other hand, it would be expected that a stronger version of Theorem 3.9 would hold. Such a strengthening would be based on the notion of minor of signed graphs rather than minor of graphs. More precisely the following conjecture is stronger than Conjecture 3.1.

Conjecture 3.5. *Given a signed graph $(G, \sigma) \in \mathcal{G}_{11} \cup \mathcal{G}_{10}$, if (G, σ) has no $(K_5, -)$-minor, then it packs.*

The idea of induction on the negative girth would work here as well. That is because if σ_1 and σ_2 are two disjoint signatures each equivalent to (G, σ), then $(G/E_1, E_2)$ is a minor of (G, σ), and if $(G, \sigma) \in \mathcal{G}_{11} \cup \mathcal{G}_{10}$, then $(G/E_1, E_2) \in \mathcal{G}_{11} \cup \mathcal{G}_{10}$.

However, the class of signed graphs with no $(K_5, -)$-minor is not a sparse family and contains signed graphs with $O(n^2)$ number of edges. Thus one cannot expect the discharging technique we used here to work directly. However, one may look for decomposition results where the planar case studied here would work as a base class.

Chapter 4

Packing antibalanced triangle-free signed planar graphs

This chapter is based on the following paper:

[40] R. Naserasr and W. Yu. On the packing number of antibalanced signed simple planar graphs of negative girth at least 5. Submitted, 2022.

As introduced in Chapter 3, the following conjecture is a reformulation of Conjecture 1.1, which is an extension of the 4-color Theorem.

Conjecture 4.1. *Every signed simple planar graph in* $\mathcal{G}_{11} \cup \mathcal{G}_{10}$ *packs.*

In the subclass of planar graphs the conjecture can restated using the dual notion of packing T-joins where T would be vertices of the dual that correspond to the negative faces of the planar embedding. The statement of the conjecture based on the notion packing T-join was first proposed by B. Guenin in early 2000's who then gave a proof of the next two cases. In our language that would be proving the conjecture for members of the class whose negative girth is 4 or 5. The T-join approach is extended in three follow up work which means that the conjecture is proved for the cases with negative girth at most 8. We note that proof for each case of girth condition relies on the proof for the earlier cases, thus dependent on the proof of the 4-color theorem. However, the work of Guenin remains unpublished and mostly not available.

An independent proof for the case of girth 4 is given in Chapter 3. This proof has extra advantage that works for any minor closed family that are 4-colorable. Thus, on the one hand it works for the larger family of K_5-minor free graphs, and, it provides a proof with the use of the 4-color theorem for subclasses such as graphs of treewidth at most 3.

In this chapter, we continue using the language of packing number and extend the technique as last Chapter to verify the case of negative girth 5 of Conjecture 4.1. More precisely, we prove that for any antibalanced signed planar graph (G, σ) of negative girth at least 5, we have $\rho(G, \sigma) \geqslant 5$. In Section 4.1, we first give some notions and then give full picture of the proof, start with providing a reformulation of the theorem and then we do a double induction and use different statements for different directions of the induction. In Section 4.2, we stablish a rich enough structure around vertices of degree 2 and 3 to apply discharging technique in Section 4.3 and finally get a contradiction with Euler's formula.

4.1 Preliminaries

Given a signed graph (G, σ) of negative girth k, a negative cycle C of it is said to be *super negative* if it has at most $k-2$ positive edges. The key property of a super negative cycle, relevant to this study, is in the following observation. Let σ' be a signature equivalent to σ but disjoint from it, one can easily find such a signature using following Theorem 4.1.

Theorem 4.1. *Given signed graph (G, σ) and a set $\sigma_1, \ldots, \sigma_r$ of signatures each equivalent to σ, there exists a signature σ' which has no common negative edge with any of (G, σ_i)'s if and only if the set $\cup_{i=1}^{r} E_i^-$ induces a bipartite graph.*

Let $G_{/\sigma}$ be the graph obtained from G by contracting the negative edges of σ and let σ' be the signature on $G_{/\sigma}$ where the negative edges of it are the images of the negative edges of (G, σ'). That $(G_{/\sigma}, \sigma')$ is well defined because the two signatures do not share a negative edge. Now a negative cycle C in $(G_{/\sigma}, \sigma')$ is of length less than or equal to $k-2$ if and only if it is the image of a super negative cycle of (G, σ). This is the key point in showing that the following is equivalent to Conjecture 4.1. We refer to Chapter 3 for more details.

Conjecture 4.2. *Any signed planar graph in $\mathcal{G}_{11} \cup \mathcal{G}_{10}$ admits an equivalent signature σ' where (G, σ') has no super negative cycle.*

We shall note that the property of having no super negative cycle is a homomorphism property in the following sense: Suppose (H, π) is a signed graph where every negative cycle has at least l positive edges. If a signed graph (G, σ) maps to (H, π), then there is a signature σ' equivalent to σ such that in (G, σ') each negative cycle has at least l positive edges. One such choice for σ' is by taking inverse image of π under the homomorphism of (G, σ) to (H, π).

This observation and Theorem 3.2 imply that given an integer k, a minimum counterexample (G, σ) of negative girth k to each of Conjecture 4.1 and Conjecture 4.2 must have no proper homomorphic image which satisfies all three conditions: It negative girth k, it is planar, and it is in $\mathcal{G}_{11} \cup \mathcal{G}_{10}$. Then, combined with the folding lemma of [29] which applies to cases in $\mathcal{G}_{11}$ and the folding lemma of [37] that applies to cases in $\mathcal{G}_{10}$, we conclude that in every planar embedding of (G, σ) each face must be a negative k-cycle.

The rest of this chapter is about proving the following theorem.

Theorem 4.2. *For any antibalanced signed simple planar graph (G, σ) of negative girth at least 5, we have $\rho(G, \sigma) \geqslant 5$.*

The following is the full picture of the proof. We are assuming that each planar graphs in $\mathcal{G}_{11} \cup \mathcal{G}_{10}$ with negative girth at most 4 packs. Let us take a planar graph (G, σ) in $\mathcal{G}_{11}$ with negative give at least 5. We want to prove that $\rho(G, \sigma) \geqslant 5$. Let us suppose we can find a switching equivalent signature σ' such that (G, σ') has no super negative cycle. By Theorem 4.1, we can find a second equivalent signature σ'' such that (G, σ') and (G, σ'') have no negative edge in common. We then contract all the negative edges in (G, σ') and consider the negative edges of σ'' as a signature on this new graph. This would be a signed planar graph in $\mathcal{G}_{10}$ whose negative cycles are of length at least 4. Applying the case of negative girth 4, we have four disjoint signatures on the contracted graphs. Together with σ' we have a total of five signature with no pair of them having a common negative edge.

So what remains is to show is that (G, σ) admits an equivalent signature with no super negative cycle. At this point the second inductive step kicks in. We assume G is a smallest counterexample. That is to say: G is a planar graph in $\mathcal{G}_{11}$ which has no loop and no triangle, it does not admits a packing of size five and among all such example, it has (first) minimum number of vertices and (second) minimum number of edges. The order on the number of vertices together with the folding lemma implies that all faces are 5-cycles. The minimality of the number of edges means removing any edge e, the remaining signed graph must admit a 5-packing. Viewing each of these five signatures as a signature on G, equivalent to σ, we must have a super negative cycle. However, each such a cycle must include e. This would be enough to stablish a rich enough structure around vertices of degree 2 and 3 to apply discharging technique and get a contradiction with Euler's formula. Thus we split details of the proof to three parts: dealing with 2-vertices, 3-vertices and then discharging.

4.2 Structural properties of the vertices

4.2.1 2-vertices

Let v be a vertex of degree 2 in G and let x and y be its two neighbours, furthermore, in the rest of this subsection e is the edge vx and e' is the edge vy.

Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ be the five signatures equivalent to σ such that, when restricted on $G - e$, they have no common negative edge. Thus e is the only potentially common negative edge among some of these signatures. Each (G, σ_i) must contain a super negative cycle. If more than one, then we choose one and name it C_i. Moreover we denote by P_i the $x - y$ path in C_i that does not contain v. Furthermore, we assume σ_i's are minimal in the sense that there is no other signature on $G - e$ equivalent to σ such that all its negative edges are also negative in σ_i. Clearly replacing each signature with a minimal one does not affect the packing property. However, then we may have a set of edges each of which is positive in all five of $(G - e, \sigma_i)$. Let E_6 be such set of edges of $G - e$. We proceed with a series of claims.

Claim 4.1. *We have one of two:*

- *Either $\sigma_i(e) \neq \sigma_i(e')$ for each i, $i = 1, 2, \ldots, 5$, in which case all the positive edges of each C_i must be in E_6.*
- *Or for exactly one of the five signatures, say σ_5, we have $\sigma_i(e) = \sigma_i(e')$ in which case the positive edge of each P_i in (G, σ_i), $i = 1, 2, 3, 4$, is a negative edge in (G, σ_5).*

Proof. First we show that we cannot have two such signatures satisfying $\sigma_i(e) = \sigma_i(e')$. Suppose to the contrary that two of them, say σ_1 and σ_2, assign the same sign to e and e'. By switching at v, if necessary, in each of (G, σ_1) and (G, σ_2) we may assume that $\sigma_1(e) = \sigma_1(e') = +$ and $\sigma_2(e) = \sigma_2(e') = +$. This implies that all the edges of P_1 are given a negative sign in (G, σ_1) and, similarly, all the edges of P_2 are given a negative sign in (G, σ_2), and thus a positive sign in (G, σ_1). Recall that, since each C_i is an odd cycle, each P_i is a path of odd length. Then the closed walk induced by $P_1 \cup P_2$, in (G, σ_1), and hence in (G, σ), is negative closed walk of even length. This contradicts the fact that $(G, \sigma) \in \mathcal{G}_{11}$.

Hence, and without loss of generality, we assume $\sigma_i(e) \neq \sigma_i(e')$ for $i = 1, 2, 3, 4$. Then for each i, $i = 1, 2, 3, 4$, the path P_i has a unique positive edge in (G, σ_i). Let us name this edge e_i. Then we first observe that e_i cannot be negative in any of (G, σ_j), $j = 1, 2, 3, 4$ as otherwise, C_i would be a positive cycle in (G, σ_j). If $\sigma_5(e) \neq \sigma_5(e')$, then for C_i, $i = 1, 2, 3, 4$, to be negative in (G, σ_5) we have $\sigma_5(e_i) = +$ which implies the first case of the claim. If $\sigma_5(e) = \sigma_5(e')$, then for C_i, $i = 1, 2, 3, 4$, to be negative in (G, σ_5) we must have $\sigma_5(e_i) = -$ in which case we have the second part of the claim. □

Note that one may change the sign of all edges in E_6 to negative in (G, σ_5). As $(G - e, \sigma)$ is in $\mathcal{G}_{11}$, the resulting signature is also equivalent to σ. Thus we may assume that the second item of the claim is always the case at the cost of allowing σ_5 not to be minimal. Under this assumption, we may also assume that $\sigma_5(e) = \sigma_5(e') = +$, as otherwise we may switch at v in (G, σ_5). This, in particular, means that for any super negative cycle C_5, e and e' are the only positive edges.

We should note that in choosing the super negative cycle C_i of (G, σ_i) one may have more than one choice. Next we aim at showing that among the possible choices, at least one should have a fair number of high degree vertices. Recall that in our case of negative girth 5 a super negative cycle has either 2 or 0 negative edges. Thus if a super negative cycle has at least one positive edges, then it has precisely two positive edges.

Claim 4.2. *Assume σ' is a minimal signature equivalent to σ such that every super negative cycle of (G, σ') contains xvy with one positive edge and one negative edge, and that, moreover, the other positive edge is incident to either x or y. Then in one of the super negative cycles of (G, σ') every vertex which is not incident to a positive edge is of degree at least 4 in G.*

Proof. That σ' is assumed to be a minimal signature implies, in particular, that no vertex is incident to only negative edges. Among all the signatures for which the conditions of Claim 4.2 hold but the conclusion does not, we take σ' to be one where the number of super negative cycles of (G, σ') is minimized. To get a contradiction we need to show that this number must be 0.

Suppose not and let $C_1, C_2, \dots C_r$ be the set of super negative cycles of (G, σ') and assume that C_1 is a shortest one among these cycles. Since the condition does not hold, C_1 has a vertex z whose two neighbours on C_1 are connected to it by negative edges (with respect to the signature σ') and $d_G(z) \leqslant 3$. Since not all edges incident to a vertex are negative, we must have $d_G(z) = 3$ and that the third neighbour of z, say z', is adjacent to it with a positive edge. We first claim that $z' \notin \{x, y\}$. Let P_1' be the $x - z$ path in C_1 which does not include v and P_1'' be the $z - y$ path which does not include v. Observe that only one of P_1' and P_1'' have a positive edge. We continue the proof assuming that P_1'' has a positive edge, which then must be incident to y. The other case would be symmetric. If $z' = x$, then P_1' together with xz induces a cycle with exactly one positive edges, depending on the parity of the length, that would either be a negative even cycle or a positive odd cycle both of which is forbidden in a member of $\mathcal{G}_{11}$. If $z' = y$, then the cycle C_1' obtained from C_1 by replacing P_1'' with the zy is also a super negative cycle of (G, σ_1) whose length is less than C_1, contradicting the choice of C_1.

Since $z' \notin \{x, y\}$, and by our assumption that in every super negative cycle of (G, σ') each positive edge is either incident to x or to y, we conclude that the edge zz' does not belong to any super negative cycle of (G, σ'). We now consider the signature σ'' obtained from (G, σ') by a switching at z. Then the each super negative cycle of (G, σ'') is also a super negative cycle of (G, σ') with the same signature. Thus (G, σ'') also satisfies the conditions of the claim, but it less super negative super cycles than (G, σ'), contradicting the choice of σ'. □

We note that each of σ_1, σ_2, σ_3, and σ_4 satisfies the conditions of the claim, and, therefore, the conclusion hold on these four signatures. For σ_5 this would depend on the possible cases of Claim 4.1. To take a better advantage of this case, we consider a signature σ_5' where the negative edges are those of σ_5 and the edges in E_6. It is already mentioned that σ_5' is an equivalent signature. We have to following claim on (G, σ_5').

Claim 4.3. *In (G, σ_5') there exist a super negative cycle C in which all vertices, but possibly x, v and y, have degree at least 4 in G.*

Proof. Observe that in (G, σ_5') the edges xv and vy are of the same sign. Thus if needed, by a switching at v we may assume they are both positive. This implies that in every super negative cycle of (G, σ_5') all edges not incident to v are negative. Let $C_1, C_2, \dots, C_r$ be the set of super negative cycles of (G, σ_5'). If each of them has a vertex of degree 2 or 3, by switching at all those vertices we will get a signature with no super negative cycle, contradicting the minimality of the counterexample. The details that such switching does not create new super negative cycles and that each switching kills of the corresponding super negative cycle is similar to the previous claim. □

Claim 4.4. *In each of (G, σ_i), $i = 1, 2, 3, 4$, one of the followings holds:*

- *Either x or y has a negative neighbour whose degree in G is at least 4.*

- *Each of x and y have a negative neighbour of degree 3.*

Proof. Suppose to the contrary that one of them, say (G, σ_1) does not satisfy the claim. That means, for one of x of y, say y, all negative neighbours (possibly none) are of degree 2. Let (G, σ_1') be obtained from (G, σ_1) by switching at all negative neighbours of y. Since each of these vertices are of degree 2 and each is incident to at least one negative edge, the switching does not create a new super negative cycle. As y has no negative neighbour in (G, σ_1'), the condition of Claim 4.2 holds for (G, σ_1'). Thus (G, σ_1') has a super negative cycle C where each vertex not incident to a positive edge is of degree at least 4. Let x' be the neighbour of x in C, $x \neq v$. Since C must be of length at least 5 and both positive edges are incident to y, both edge of C incident with x' are negative and thus x' has degree at least 4. Moreover, as x is not adjacent to y, and switchings were done only at neighbours of y, the sign of the edge xx' is negative in (G, σ_1) as well. This means x' is a negative neighbour of x whose degree is at least 4, thus the first case of the claim holds. □

Claim 4.5. *Suppose that u and v are two adjacent 2-vertices with u' and v' being the other neighbour, respectively. Then both u' and v' have degree at least 6 and have at least 5 4^+-neighbours.*

Proof. By minimality of the counterexample we have signature packing $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ of $(G - \{u, v\}, \sigma)$. Each of these signatures can be extended to G such that first of all (G, σ_i) is equivalent to (G, σ), secondly in each of them both uv and vv' are positive, and thirdly, the condition that uv and vv' having the same sign can be fulfilled by switching, if necessary, at v, u or both.

Since each (G, σ_i) has to have a super negative cycle, then uu' must be a negative edge in all of them and this would be the only common negative edge between any pair of them. Each of these five signatures, however, satisfies the conditions of Claim 4.1, thus there is a super negative cycle C_i in (G, σ_i) where vertices not incident to positive edges are 4^+-vertices. In C_i the neighbour u_i of u', $u_i \neq u$, is not incident to a positive edge. Since $u'u_i$ is negative only in (G, σ_i), the vertices u_i are 5 distinct 4^+-neighbours of u'. As u is also a distinct neighbour of u', it has total of at least six neighbours. The claim for v' follows by symmetry. □

Claim 4.6. *Suppose that u is a 2-vertex with u' and v as neighbours and that v is 3-vertex with its two other neighbours being v_1 and v_2. Then, first of all, u' has at least four 4^+-neighbours. Secondly, among v_1 and v_2 either one has at least four 4^+-neighbours or together they have at least five 4^+-neighbours.*

Proof. We consider induced signed subgraph by deleting the edge uu' and as before define σ_1, σ_2, σ_3, σ_4 to be four minimal signatures with no common negative edge and let σ_5' be the signature which assigns negative to the edges that are not negative in any of $(G - uu', \sigma_i)$, $i \leqslant 4$. As before, we consider σ_i and σ_5' as signatures on G rather than $G - uu'$, thus some of them have uu' as (the only) common negative edge.

By our choice of σ_5' only the second case of Claim 4.1 can happen. Then if necessary, in (G, σ_5') we switch at u to get a (G, σ_5'') where uu' and uv are both positive, noting that each super negative cycle of (G, σ_5'') is also a super negative cycle of (G, σ_5'). As there must be at least one such cycle, and as there are already two positive edges, all other edges must be negative. That implies that, in particular, at least one of the two edges vv_1 and vv_2 is negative in (G, σ_5'). We consider two cases depending on if only one is negative or both.

First assume the case that $\sigma_5'(vv_1) = -$ and $\sigma_5'(vv_2) = +$. Since each edge beside uu' is negative in only one of the signatures, we may assume $\sigma_i'(vv_1) = -$. Then for each $j \neq i$, in (G, σ_i) all the positive edges of each of the super negative cycle are incident to v, and, thus, by Claim 4.2 for $j \leqslant 4$ and by Claim 4.3 in the case of $j = 5$ we have a super negative cycle in (G, σ_j) in which the neighbour of u' distinct from u is of degree at least 4. We note moreover that the positive edges of any super negative cycle in (G, σ_j) are negative in σ_5'. This implies that vv_2 cannot be a positive edge in these cycles. Thus the second positive edge of any super negative cycle in (G, σ_j), $j \leqslant 4$, $j \neq i$ is vv_1. Again using Claim 4.2 the neighbour of v_1 in each of these cycles must be at least of degree 4. Since that is the case for the super negative cycle of (G, σ_5') as well, v_1 must have at least four such neighbours.

Now we consider the case that $\sigma_5'(vv_1) = \sigma_5'(vv_2) = -$. In this case then for all j's, $j = 1, 2, \ldots, 5$ every super negative has two positive edges incident with v. Thus, first of all u' will have at least five 4^+-neighbours, secondly, each of the signatures will imply a 4^+-neighbour for either v_1 or for v_2, giving a total of at least five such neighbours for the two of them. $\square$

4.2.2 3-vertices

Similar to the last subsection, let $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5'$ be the five signatures equivalent to σ such that, when restricted on $G - e$, for a fixed edge e, each edge in $G - e$ is negative in exactly one of these five signatures, and σ_i's are minimal for $i = 1, 2, 3, 4$. Thus e is the only potentially common negative edge among some of these signatures. As (G, σ_i) is a counterexample to Conjecture 4.2, each (G, σ_i) contains at least one super negative cycle, one of which is named C_i.

Claim 4.7. *Every 4-cycle of G contains a vertex of degree at least 4.*

Proof. Suppose not, let $C = v_1v_2v_3v_4v_1$ be a 4-cycle that all its vertices have degree at most 3. By the folding lemma, every face of G is of length 5. Thus C is not a facial cycle, hence it is a separating cycle. Moreover, since G is 2-connected, at least two of v_1, v_2, v_3, v_4 have neighbours inside of C, and similarly at least two of them have neighbours outside. But since each v_i is a 3^--vertex, it follows that they are all 3-vertices and that precisely two of them have neighbours inside and two of them have neighbours outside. By symmetry, we consider two case: (1) v_1, v_2 have neighbours inside C, (2) v_1, v_3 have neighbours inside C. In case (1), the path $v_1v_4v_3v_2$ is part of a facial cycle inside C. As every facial cycle is a 5-cycle, there is a common neighbour of v_1 and v_2. But that would make triangle with v_1v_2. In case (2), considering the faces inside C formed by $v_1v_2v_3$ and $v_1v_4v_3$, we conclude that the neighbours x, y of v_1 and v_3 inside C are themself adjacent and that the edge xy is part of both mentioned faces. That implies that x and y are adjacent 2-vertices. But we have already seen that for adjacent 2-vertices x, y their other neighbours must be of degree at least 6. □

Claim 4.8. *If C is a shortest super negative cycle, then C contains no chord.*

Proof. Observe that a chord on a negative cycle creates one positive cycle and one negative cycle. Let C be a shortest super negative cycle with a chord e. Let C' be the negative cycle created by C and e. We claim that C' is a shorter super negative cycle, contradicting the choice of C. That C' is negative is by our choice. That it is shorter is by the fact that there are no parallel edges and e is a chord of C. It remains to show that C' is super negative, i.e. it has at most two positive edges. Since C has at most two positive edges in $C \cup \{e\}$ there are at most three positive edges. But as (G, σ) is switching equivalent to $(G, -)$, every negative cycle (which is an odd cycle of G) has an even number of positive edges, thus C' has at most two positive edges. □

Claim 4.9. *Let v be a vertex of degree 3 in G and $N(v) = \{v_1, v_2, v_3\}$, such that both v_2 and v_3 have degree 3. Let σ' be a signature equivalent to σ such that every super negative cycle of (G, σ') contains vv_1, noting that such a signature exists by the minimality of (G, σ). If (G, σ') has the extra property that every super negative cycle has two positive edges each of which is incident to at least one of v, v_2 or v_3, then there exists a super negative cycle $C_{\sigma'}$ such that every vertex not incident to a positive edge is of degree at least 4 in G.*

Proof. Among all the signatures for which the conditions of Claim 4.9 hold but the conclusion does not, we take σ' to be one where the number of super negative cycles of (G, σ') is minimum. To get a contradiction we would like to show that this number must be 0. Let $N(v_i) = \{v, x_i, y_i\}$ for $i = 2, 3$.

Let $C_1, C_2, \ldots, C_r$ be the set of super negative cycles of (G, σ') and assume that C_1 is a shortest one among these cycles. Since the conclusion of the claim on (G, σ') does not hold, C_1 has a vertex z whose two neighbours on C_1 are connected to it by negative edges (with respect to the signature σ') and $d_G(z) \leqslant 3$. If all edges incident to z are negative, then we consider (G, σ'') obtained from (G, σ) by switching at z. We observe that super negative cycles of (G, σ'') are exactly those super negative cycles of (G, σ') which do not contain z. Thus (G, σ'') also satisfies the conditions of the claim, but it has less super negative super cycles than (G, σ'), contradicting the choice of σ'.

Since both edges of C_1 incident to z are negative we must have $d_G(z) = 3$ and that the third neighbour of z, say z', is adjacent to it with a positive edge. We claim zz' belongs to some super negative cycle of (G, σ'). Suppose not. Let π be the signature obtained from (G, σ') by switching at z. Then, first of all, there is still no super negative cycle in (G, π) containing zz', because for cycles containing this edge number of positive edges is the same in (G, σ') and (G, π). Secondly, any super negative cycle of (G, σ') containing z has two more positive edges in (G, π). Since we assume every super negative cycle of (G, σ') has two positive edges, those containing z, in particular C_1, are not super negative in (G, π). This contradicts with the number of super negative cycles of (G, σ') being minimum. Thus zz' is in a super negative cycle, say C_i, $2 \leqslant i \leqslant r$.

Next we claim that $z \notin \{v, v_1\}$. We assume to contrary and first consider the case that $z = v$. Recall that vv_1 is an edge of C_1. Between v_2 and v_3, by symmetry, assume $vv_3 \in C_1$. As edges of C_1 incident to z are negative we have $\sigma'(vv_1) = \sigma'(vv_3) = -$ and since not all edges incident to z are negative we have $\sigma'(vv_2) = +$. Since C_1 must have two positive edges, and they must be incident to v or v_2 or v_3, the vertex v_2 should be on C_1 and moreover should be incident to a positive edge of C_1. Noting that vv_2 is not an edge of C_1, $x_2v_2y_2$ should be a part of C_1. This implies that vv_2 is a chord of C_1, contradicting Claim 4.8. Next we consider the case that $z = v_1$. Recall that zz' is a positive edge of a super negative cycle C_i of (G, σ') for some $2 \leqslant i \leqslant r$. By the assumption, every positive edge of C_i is incident to one of v, v_2 or v_3. That means v_1 is adjacent to one of v, v_2, v_3 with a positive edge. If vv_1 is a positive edge we have a digon otherwise we have a triangle. But either case contradicts the assumption on (G, σ).

Since zz' must be a positive edge of super negative cycle C_i and since all such edges are incident to one of v, v_2, v_3 we must have $z \in \{v_2, v_3, x_2, y_2, x_3, y_3\}$. By symmetries we consider only two possibilities of $z = v_2$ or $z = x_2$. First let $z = v_2$. If $z' = v$, then C_1 contains the edge zz' as a chord and we have contradiction with Claim 4.8. If $z' \in \{x_2, y_2\}$, say $z' = y_2$, then $\sigma'(vv_2) = \sigma'(v_2x_2) = -$, since vv_1 is also an edge of C_1, vv_3 is not. As all positive edges are incident to v, v_2 or v_3 and since there are two such edges in C_1, v_3 is a vertex of C_1, but then again vv_3 is a chord contradicting Claim 4.8.

Finally assume $z = x_2$ and let $N(x_2) = \{v_2, x_2', x_2''\}$. If $z' = x_2'$ or $z' = x_2''$, since zz' belongs to a super negative cycle, positive edges of super negative cycles are incident to v, v_2 or v_3 and as G contains no triangle, $z' = v_3$, in which case $vv_2x_2v_3v$ is a 4-cycle which contains four 3-vertices, contradicting Claim 4.7. Therefore, we must have $z' = v_2$, then $\sigma'(x_2x_2') = \sigma'(x_2x_2'') = -$ and $x_2'x_2x_2''$ is a part of C_1. If $vv_2 \in C_1$, then zz' is again a chord of C_1, which contradicts Claim 4.8. So $vv_2 \notin C_1$, by symmetry we may write C_1 as $v_1vv_3x_3P_1x_2'x_2x_2''P_2v_1$. If $\sigma'(vv_2) = -$, then again vv_2x_2 creates two cycles from C_1 one of which is negative. And this negative cycle has at most two positive edges, therefore is a super negative cycle and thus contains the edge vv_1. By Claim 4.7, path $v_1vv_3x_3P_1x_2'x_2$ must have length at least 3, as otherwise together with v_2 we will have a 4-cycle all whose vertices are of degree 3. Replacing this path with vv_2x_2 we find a shorter super negative cycle, contradicting the minimality of C_1. Next let $\sigma'(vv_2) = +$. By the assumption on C_1, the fact that every cycle has even number of positive edges, and the fact that the edges of the path $x_2x_2''P_2v_1$ are all negative, we must have $\sigma'(vv_1) = -$, as otherwise the cycle $v_1vv_2x_2x_2''P_2v_1$ has three positive edges. Since positive edges of C_1 must be incident to v, v_2 or v_3, we have $\sigma'(vv_3) = \sigma'(v_3x_3) = +$. Furthermore $x_2''P_2v_1$ has an even number of edges since otherwise $v_1vv_2x_2x_2''P_2v_1$ is a shorter super negative cycle. Recall that zz' is also in the super negative cycle C_i. We consider the following two cases.

1. If $vv_2 \in C_i$, then by our assumption $vv_3 \notin C_i$, thus we may write C_i as $v_1vv_2x_2P_3v_1$, but then the cycle obtained from two paths from x_2 to v_1 of C_1 and C_i forms a super negative cycle which does not contain vv_1 and contains no positive edge, a contradiction.

2. Otherwise $vv_2 \notin C_i$, let $C_i = v_1vv_3y_3P_3x'_2x_2v_2y_2P_4v_1$. But then the cycle $vv_3y_3P_3x'_2$ x_2v_2v contains exactly three positive edges, which never happens in (G, σ'). □

Claim 4.10. *Let v be a vertex of degree 3 in G and $N(v) = \{v_1, v_2, v_3\}$, such that both v_2 and v_3 have degree 3. Then v_1 has at least two neighbours of degree at least 4.*

Proof. Let $G' = G - vv_1$ and $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma'_5$ be the five signatures equivalent to σ such that, when restricted on G', they have no common negative edge. Thus vv_1 is the only potentially common negative edge among some of these signatures. Each (G, σ_i) must contain a super negative cycle using the edge vv_1. If more than one, then we choose one and name it C_i. Let $N(v_i) = \{v, x_i, y_i\}$ for $i = 2, 3$.

We first claim that among all these five signatures, there are at least two, in which, after switching at v, v_2 and v_3 (if necessary), we have the following: first of all in each of the four paths $v_1vv_ix_i$ and $v_1vv_iy_i$, $i = 2, 3$, there are at least two positive edges, secondly, we do not create any new super negative cycle. First suppose vv_2 and vv_3 belong to the same signature, say E_1^-. Then in each of the other signatures, namely $\sigma_2, \sigma_3, \sigma_4$ and σ'_5, vv_2 and vv_3 are both positive. If vv_1 is also positive on at least two of $\sigma_2, \sigma_3, \sigma_4, \sigma'_5$, then we are done. Assume in (G, σ_2), vv_1 is negative and both vv_2 and vv_3 are positive. Let σ'_2 be obtained by switching at v. We first observe that every super negative cycle of (G, σ'_2) is also a super negative cycle of (G, σ_2) because for $i = 2, 3$, if one of v_ix_i or v_iy_i is negative in σ'_2, then we switch at v_i. Let σ''_2 be the resulting signature. We observe that first of all no new super negative cycle is created, secondly in (G, σ''_2) each of $v_1vv_ix_i$ and $v_1vv_iy_i$ has at least two positive edges. Next w.l.o.g. let $vv_2 \in E_1^-$ and $vv_3 \in E_2^-$. Then in (G, σ_i), $i = 1, 2$, we could first switch at v (if necessary) to make vv_1 positive, since exactly one of vv_2 and vv_3 is positive, this switching will not create new super negative cycle. And then similarly we could either switch at v_2 or v_3 if necessary.

Therefore, we could find σ'_1 and σ'_2 obtained from σ_1 and σ_2 by switching at v, v_2 and v_3, such that all the super negative cycles use the edge vv_1 and their two positive edges are either incident to v, v_2 or v_3. Thus by Claim 4.9, their exists a super negative cycle C'_1 of (G, σ'_1) (similarly in (G, σ'_2)) such that every vertex not incident to a positive edge is of degree at least 4 in G. Let v'_1 be another neighbour of v_1 in C'_1, since v_1 is not adjacent to v_2 or v_3, $v_1v'_1 \in \sigma_1$. Then we claim that $d(v'_1) \geqslant 4$. If not, v'_1 is incident to a positive edge of C'_1, which means $v'_1 \in \{x_2, y_2, x_3, y_3\}$. Let $v'_1 = x_2$, since each super negative cycle has length at least 5 and C'_1 uses the edge vv_1, we have $vv_2 \notin C'_1$ and $x_2v_3 \notin C'_1$ if it exists. Therefore both v_2x_2 and v_2y_2 belong to C'_1 and v_2x_2 is positive. Since $vv_3 \in C'_1$, by our claim above C'_1 contains at least three positive edges, a contradiction. Therefore $d(v'_1) \geqslant 4$. Together with the same conclusion in (G, σ'_2), we have that v_1 has at least two neighbours of degree at least 4. □

Claim 4.11. *Let u and v be two adjacent vertices of degree 3 in G. Assume σ' is a signature equivalent to σ, such that every super negative cycle of (G, σ') contains uv and contains two positive edges which are incident to either u or v. Then there exists a super negative cycle such that every vertex not incident to a positive edge is of degree at least 4 in G.*

Proof. As in the proof of Claim 4.9 among all the signatures for which the conditions of Claim 4.11 hold but the conclusion does not, we take σ' to be one where the number of super negative cycles of (G, σ') is minimum, and, moreover, we take σ' to be a minimal signature. Let $N(u) = \{u_1, u_2\}$ and $N(v) = \{v_1, v_2\}$.

Let $C_1, C_2, \ldots, C_r$ be the super negative cycles of (G, σ'), and assume w.l.o.g. that $|C_1| \leqslant |C_j|$, $2 \leqslant j \leqslant r$. If the conclusion does not hold, then C_1 has a vertex z whose two neighbours on C_1 are connected to it by negative edges and $d_G(z) \leqslant 3$. Minimality of σ' implies that $d_G(z) = 3$ and that the third neighbour of z, say z', is adjacent to it with a positive edge. Furthermore, zz' is in a super negative cycle, say C_i, $2 \leqslant i \leqslant r$, as otherwise by switching at z we have less super negative cycles.

As each of u and v is incident to a positive edge of C_1, $z \notin \{u, v\}$. Considering the super negative cycle C_i, zz' is a positive edge, thus by our assumption one of the end point is u or v. As $z \notin \{u, v\}$, we have $z' \in \{u, v\}$ and hence $z \in \{u_1, u_2, v_1, v_2\}$. W.l.o.g. let $z = u_1$ and $z' = u$. But then uu_1 is a chord of C_1 and we have a contradiction with Claim 4.8. □

Claim 4.12. *Let v_1, v_2, v_3 and v_4 be four vertices of degree 3, and σ' be a signature equivalent to σ such that the following holds.*

1. v_i is adjacent to v_{i+1}, $i = 1, 2, 3$.

2. $\sigma'(v_1v_2) = -$, $\sigma'(v_2v_3) = \sigma'(v_3v_4) = +$.

3. Each of v_2 and v_3 is incident to exactly two positive edges.

4. Either v_1 is incident to two positive edges or v_4 is incident to three positive edges.

5. Every super negative cycle of (G, σ') contains the positive edge v_2v_3.

6. The other positive edge of any other super negative cycle must be incident to one of the v_i ($i \in \{1, 2, 3, 4\}$).

Then there exists a super negative cycle of (G, σ') such that every vertex not incident to a positive edge is of degree at least 4 in G.

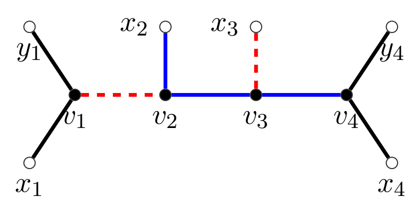

Figure 4.1: Four vertices of degree 3 in Claim 4.12

Proof. Again among all the signatures for which the conditions of Claim 4.12 hold but the conclusion does not, we take σ' to be one where the number of super negative cycles of (G, σ') is minimum. Let the other two neighbours of v_i be x_i, y_i for $i = 1, 4$, and the third neighbour of v_j be x_j for $j = 2, 3$, as shown in Figure 4.1. Suppose to the contrary that $C_1, C_2, \ldots, C_r$ are the set of super negative cycles of (G, σ') and assume that C_1 is a shortest one among these cycles. Thus C_1 has a vertex z whose two neighbours on C_1 are connected to it by negative edges and $d_G(z) \leqslant 3$. It follows that $d_G(z) = 3$ and that the third neighbour of z, say z', is a positive neighbour, moreover, zz' is in a super negative cycle, say C_l, $2 \leqslant l \leqslant r$.

Since in (G, σ') every super negative cycle contains v_2v_3 as a positive edge, $z \notin \{v_2, v_3\}$. And since each positive edge of any super negative cycle is incident to some v_i, we have $z \in \{v_1, v_4, x_1, x_2, x_3, x_4, y_1, y_4\}$. By symmetries we consider following possibilities.

1. $z = v_1$. Since $\sigma'(v_1v_2) = -$, $v_1v_2 \in C_1$ and at least one of v_1x_1, v_1y_1 is negative. By assumption v_4 is incident to three positive edges, and thus $v_4 \notin C_1$. This contradicts with the fact that positive edges of every super negative are incident to v_i.

2. $z = v_4$. Since $\sigma'(v_3v_4) = +$, and edges in C_1 incident to z are both negative, we have that $v_4x_4, v_4y_4 \in C_1$ and $\sigma'(v_4x_4) = \sigma'(v_4y_4) = -$, and hence $z' = v_3$. By the original assumption of the claim, $\sigma'(v_1x_1) = \sigma'(v_1y_1) = +$. Recall that every super negative cycle of (G, σ') must contain v_2v_3. But any cycle that contains both v_2v_3 and v_3v_4 must contain at least one more positive edge. This is contradiction with the fact that zz' is in a super negative cycle.

3. $z = x_2$. Since $\sigma'(v_2x_2) = +$ and $v_2x_2 \notin C_1$ we must have $v_1v_2 \in C_1$. But then v_2x_2 is a chord of C_1 which contradicts the Claim 4.8.

4. $z = x_3$. Since the super negative cycle C_l contains the positive edges zz' and v_2v_3, it contains no other positive edges, in particular $v_2x_2 \notin C_l$. Thus $v_1 \in C_l$. So each of v_1 and x_3 is incident with at least two negative edges, and they are not connected by a negative edge, since otherwise $v_1v_2v_3x_3$ induces a negative 4-cycle. Recall that if a vertex of degree 3 is incident with at least two negative edges, then a switching at it may eliminate some super negative cycle, but will never create a new one. Thus if we switch at both v_1 and x_3, then the remaining set of super negative cycles all must still contain the edge v_2v_3. But then in the new signature all edges incident to v_2 and v_3 are positive, which implies that every cycle containing v_2v_3 has at least three positive edges and there can be no super negative cycle.

5. $z = x_4$. First suppose $v_4x_4 \in C_1$, then $\sigma'(v_4x_4) = -$ and by the assumption $\sigma'(v_1x_1) = \sigma'(v_1y_1) = +$. Therefore $v_3v_4 \notin C_1$, since otherwise there will be three positive edges in C_1. However in this case v_3v_4 is a chord of C_1, this contradicts the Claim 4.8. Now suppose $v_4x_4 \notin C_1$, then $z' = v_4$ and $\sigma'(v_4x_4) = +$. We now consider the super negative cycle C_l containing v_4x_4 $(= zz')$. As it must contain the positive edge v_2v_3 as well, it can have no other positive edge. In particular, v_3v_4 is not in C_l. This implies that first of all $v_4y_4 \in C_l$ and, secondly, that v_4y_4 is a negative edge in (G, σ'). But then, by the assumption of $\sigma'(v_1x_1) = \sigma'(v_1y_1) = +$, the cycle C_l contains three positive edges, contradiction with C_l being a super negative cycle because it must contain at least one of v_1x_1, v_1y_1 and v_2x_2, all of whom are positive.

6. $z = x_1$. If $v_1x_1 = v_1z$ is in C_1, then it must be a negative edge of C_1. Thus v_1 is incident to at most one positive edge. The assumption of the claim implies that all edges incident to v_4 are positive. That implies $v_4 \notin C_1$ as otherwise C_1 will have at least three positive edges. As each positive edge of C_1 should be incident to one of v_1, v_2, v_3, the second positive edge of C_1 can only be either v_1y_1 or v_2x_2, in either case it follows that v_1v_2 is a chord of C_1, a contradiction with Claim 4.8.

   So we may assume $v_1x_1 \notin C_1$. This implies that $z' = v_1$ and that $\sigma'(v_1x_1) = +$. As C_1 has no chord, $v_1 \notin C_1$. This implies $v_2x_2 \in C_1$ and since $\sigma'(v_2x_2) = +$, it is the only other positive edge of C_1. Thus $v_3v_4 \notin C_1$ and hence, $v_3x_3 \in C_1$. We now claim that $x_1x_2 \in C_1$, that is because otherwise in the union of C_1 and the path $x_1v_1v_2$ we will find a shorter super negative cycle. Moreover, we observe that $\sigma'(x_1x_2) = -$. We now consider the cycle C_1' obtained from C_1 by replacing $x_1x_2v_2$ with $x_1v_1v_2$. This cycle is also a super negative cycle of (G, σ') and is of the same length as C_1. Thus there must be a vertex z_1 of C_1' which is of degree three in G, and both edges of C_1' incident to z_1 are negative. Repeating the same argument as in cases 1-5, we conclude that $z_1 \in \{x_1, y_1\}$. The case $z_1 = y_1$ is not possible as otherwise C_1' contains a chord, and the case $z_1 = x_1$ is not possible because $\sigma'(v_1x_1) = +$. □

Claim 4.13. *Let v_1, v_2, v_3, v_4 be vertices of degree 3 in G such that v_i is adjacent to v_{i+1}, $i = 1, 2, 3$, where other neighbours of v_i's are labelled as in Figure 4.1. Then either each of x_2 and x_3 has at least three neighbours of degree at least 4, or one of x_2 and x_3 has at least four neighbours of degree at least 4.*

Proof. Let $G' = G - v_2v_3$ and let $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ be the five signatures equivalent to σ such that, when restricted on G', they have no common negative edge. So v_2v_3 can be the only common negative edge among some of these signatures. Each (G, σ_i) must contain a super negative cycle using the edge v_2v_3, one of which is named C_i.

In each signature σ_i, $i = 1, \ldots, 5$, if v_2x_2 and v_1v_2 have the same sign, then by switching at v_2 and v_3 (if necessary), we can either be sure that all the super negative cycles have exactly two positive edges and that each of them is incident to either v_2 or v_3. Then by Claim 4.11, there exists a super negative cycle C_1 such that every vertex not incident to a positive edge is of degree at least 4 in G. It is easy to observe that in at least three of the signatures $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$, say σ_1, σ_2, and σ_3, the edges v_2x_2 and v_1v_2 have the same sign. Since we only switch at v_2 and v_3, in each of the signatures σ_1, σ_2, and σ_3, either v_1 or x_2 has a negative neighbour of degree at least 4 in each of (G, σ_1), (G, σ_2), and (G, σ_3). If in each of (G, σ_4) and (G, σ_5) either the pair v_1v_2 and v_2x_2 have the same sign or the pair v_3v_4 and v_3x_3 have the same sign, then in total v_1 and x_2, as well as v_4 and x_3 have five neighbours that are of degree at least 4 in G. As either v_1 or v_4 can have at most two such neighbours, x_2 and x_3 must have at least 3 of them. We note that the conclusion holds.

Hence we suppose $\sigma_4(v_1v_2) = -\sigma_4(v_2x_2)$ and $\sigma_4(v_3v_4) = -\sigma_4(v_3x_3)$. Since we can switch at either v_2 or v_3 (if necessary), we assume v_2v_3 is positive. If $\sigma_4(v_2x_2) = \sigma_4(v_3x_3)$, then we first make v_1v_2 and v_3v_4 to be negative by switching at v_2 and v_3 (if necessary).

If at least one of v_1x_1 and v_1y_1 (resp. v_4x_4 and v_4y_4) is negative, then after switching at v_1 (resp. v_4), we will not create any new super negative cycle. Otherwise, both v_1x_1 and v_1y_1 (resp. v_4x_4 and v_4y_4) are positive. In either case, each cycle containing v_2v_3 has at least three positive edges, which is a contradiction. Therefore, we may suppose $\sigma_4(v_2x_2) = -\sigma_4(v_3x_3)$, and w.l.o.g. assume $\sigma_4(v_2x_2) = +$. By switching at v_1, if necessary, we can make sure that v_1 is incident to at least two positive edges, let the obtained signature be σ'_4. Then the positive edges of each super negative cycle in (G, σ'_4) must be incident to either v_1 or v_2. Since $\sigma'_4(v_3v_4) = +$, every super negative cycle of (G, σ'_4) contains the edge v_3x_3. By Claim 4.12, there exists a super negative cycle such that every vertex not incident to a positive edge is of degree at least 4. Therefore, either x_3 has a negative neighbour (in (G, σ_4)) of degree at least 4, or x_3 is adjacent to one of x_1 and y_1. If we switch at v_2 and v_3, by symmetry, we have that either x_2 has a negative neighbour (in (G, σ_4)) of degree at least 4, or x_2 is adjacent to one of x_4 and y_4. Now it suffices to consider two cases based on σ_5.

Case 1: Either $\sigma_5(v_1v_2) = \sigma_5(v_2x_2)$ or $\sigma_5(v_3v_4) = \sigma_5(v_3x_3)$. Applying the same argument as conclude that for each (G, σ_i), $i = 1, 2, 3$, either v_1 or x_2 has a negative neighbour of degree at least 4. Similarly either v_4 or x_3 have a negative neighbour of degree at least 4. Since $d(x_1) = d(x_4) = 3$, both x_2 and x_3 have at least two neighbours of degree at least 4. Suppose the conclusion of the claim does not hold, assume x_2 has at most two neighbours of degree at least 4, w.l.o.g. assume x_2 is adjacent to x_4. Since x_3 can have at most three neighbours of degree at least 4, x_3 is adjacent to either x_1 or y_1, which implies x_2 has at least three neighbours of degree at least 4, a contradiction.

Case 2: $\sigma_5(v_1v_2) = -\sigma_5(v_2x_2)$ and $\sigma_5(v_3v_4) = -\sigma_5(v_3x_3)$. Applying the same argument as for σ_4, either x_2 has a negative neighbour (in (G, σ_5)) of degree at least 4, or x_2 is adjacent to one of x_4 and y_4. Similarly either x_3 has a negative neighbour (in (G, σ_5)) of degree at least 4, or x_3 is adjacent to one of x_1 and y_1. Again we suppose the conclusion does not hold, and assume x_2 has at most two neighbours of degree at least 4. W.l.o.g. let x_2 be adjacent to x_4 and $\sigma_4(x_2x_4) = -$. Since x_3 can have at most three neighbours of degree at least 4, w.l.o.g. we assume x_3 is adjacent to x_1 and $\sigma_4(x_1x_3) = -$. Therefore, both x_2 and x_3 have at least two neighbours of degree at least 4. Hence, it must be the case that x_2 is adjacent to y_4 and $\sigma_5(x_2y_4) = -$, which implies that x_3 has three neighbours of degree at least 4. But then x_3 must be adjacent to y_1 and $\sigma_5(x_3y_1) = -$, which implies x_2 has three neighbours of degree at least 4, a contradiction. □

4.3 Discharging procedure

In the following, we will use the discharging technique to get a contradiction. The *initial charge* ω on $V(G) \cup F(G)$ is defined as follows: $\omega(x) = d(x) - 4$ for every $x \in V(G) \cup F(G)$. By the relation $\sum_{v \in V(G)} d(v) = \sum_{f \in F(G)} d(f) = 2|E(G)|$ and Euler's formula, the initial total charge of the vertices and faces satisfies the following:

$$\sum_{x \in V(G) \cup F(G)} \omega(x) = \sum_{x \in V(G) \cup F(G)} (d(x) - 4) = -4|V(G)| + 4|E(G)| - 4|F(G)| = -8.$$

Since any discharging procedure preserves the total charge of G, after applying appropriate discharging rules to change the initial charge ω to the final charge ω^* such that $\omega^*(x) \geqslant 0$ for every $x \in V(G) \cup F(G)$, we can have the contradiction below:

$$0 \leqslant \sum_{x \in V(G) \cup F(G)} \omega^*(x) = \sum_{x \in V(G) \cup F(G)} \omega(x) = -8,$$

and thus completes the proof.

For brevity, we call a 4^+-vertex *big*, and a 3^--vertex *small*. For a vertex v, by $n_k(v)$ we denote the number of k-neighbours of v and by $n_b(v)$ the number of big neighbours of v. Given a face f, $n_k(f)$ is the number of k-vertices incident to f. For $x, y \in V(G) \cup F(G)$, let $\tau(x \to y)$ denote the charge transferring from x to y. For a 5-face $f = [v_1 v_2 \ldots v_5 v_1]$, if it is incident to exactly two small vertices, say $d(v_1) = d(v_3) = 2$, $n_b(v_2) = 3$ and v_2 is not adjacent to any 3-vertex, then we call f and v_2 *special*. We will do discharging in three stages. Below are our needed discharging rules for first stage:

(R1) Let $d(v) \geqslant 5$. If $n_b(v) \geqslant 4$, then v sends 1 to each adjacent small vertex. Otherwise if $n_3(v) + n_b(v) \geqslant 4$, then v sends 1 to each 2-neighbour, and $\frac{d(v)-4-n_2(v)}{n_3(v)}$ to each 3-neighbour.
(R2) Let $d(f) = 5$. If $n_{3^-}(f) = 1$, then f sends 1 to the incident small vertex.

After the first round of discharging, each 3-vertices which is adjacent to a 5^+-vertex v with $n_b(v) \geqslant 4$ or incident to a face f with $n_{3^-}(f) = 1$, has a non-negative charge. If a 2-vertex is incident or adjacent to at least two of the following, then it would end up with a non-negative charge: face with only one small vertex or 5^+-vertex with four 3^+-neighbours. We call these small vertices *rich*. In the following rules, if not specified, the small vertices that we consider are those who remain negative, and refer to them as *poor* vertices. We use 5^i-face to denote 5-face incident to i poor vertices. A 3-vertex is called $3_{k,l}$-vertex, if it is adjacent to k vertices each of which has at least three big neighbours, at least two either rich or poor 3-neighbours, and is incident to l 5^2-faces.

(R3) For the 5^+-vertex v such that $n_b(v) \leqslant 3$ and $n_3(v) + n_b(v) \leqslant 3$, each of them sends $\frac{d(v)-4}{n_2(v)}$ to each 2-neighbour.
(R4) Suppose f is a non-special 5-face. Then
  (R4.1) If f is a 5^1-face, f sends 1 to incident small vertex;
  (R4.2) If f is a 5^2-face, then f sends $\frac{1}{2}$ to each small vertex incident to f.
  (R4.3) If f is a 5^3-face then
  (R4.3.1) If $n_2(f) = 2$ and $n_3(f) = 1$, then f sends $\frac{1}{2}$ to each incident 2-vertex.
  (R4.3.2) If $n_2(f) = 1$ and $n_3(f) = 2$, then f sends $\frac{1}{2}$ to the incident 2-vertex. First suppose f is incident to a $3_{k,l}$-vertex. If $k + l \geqslant 2$, then f sends $\frac{1}{2}$ to the other 3-vertex; If $k = 1$ and $l = 0$, then f sends $\frac{1}{6}$ to $3_{1,0}$-vertex and $\frac{1}{3}$ to the other 3-vertex; If $k = 0$ and $l = 1$, and moreover it is incident to a 5^3-face which contains no 2-vertex, then f sends $\frac{1}{6}$ to this $3_{0,1}$-vertex, and $\frac{1}{3}$ to the other 3-vertex. Otherwise, f sends $\frac{1}{4}$ to each incident 3-vertex.

(R4.4) If f is a 5^3-face such that $n_3(f) = 3$ or a 5^{4^+}-face, then f does not give charge to any incident $3_{k,l}$-vertex such that $k + l \geqslant 2$, but sends $\frac{1}{6}$ to each incident $3_{1,0}$-vertex, then distribute its remaining charge equally among the other incident 3-vertices.

Given a special face F with a special vertex v, a charge pot at (F, v) is the set of consecutively adjacent special faces whose special vertex is v, as shown in Figure 4.2. The total charge of the pot is the number of faces it contains as shown in (R5). After carrying out (R1)-(R4), we apply (R5) as follows.

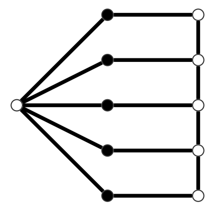

Figure 4.2: Charge pot

(R5) Special face contributes 1 to its charge pot, each 2-vertex in exactly one charge pot take needed charge such that its final charge is non-negative from its charge pot, and each 2-vertex in two different charge pots takes charge which together with the charge get from the special vertex is exactly one from each of its charge pot with respect to the special vertex.

First, we observe that the following facts are true.

Fact 4.1. *Let $d(v) = 2$ and $N(v) = \{v_1, v_2\}$, then $n_b(v_1) + n_b(v_2) + \frac{n_3(v_1)+n_3(v_2)}{2} \geqslant 6$.*

Fact 4.2. *A non-special 5-face sends charge at least $\frac{1}{2}$ to its incident 2-vertex.*

In what follows, we are going to show that $\omega^*(x) \geqslant 0$ for all $x \in V(G) \cup F(G)$ and the charge pot is also non-negative.

Let $v \in V(G)$. First suppose $d(v) \geqslant 5$. If $n_b(v) \geqslant 4$, then $\omega^*(v) \geqslant d(v) - 4 - (d(v) - 4) = 0$ by (R1). Or if $n_3(v) + n_b(v) \geqslant 4$, then $\omega^*(v) \geqslant d(v) - 4 - n_2(v) - n_3(v) \times \frac{d(v)-4-n_2(v)}{n_3(v)} = 0$ by (R1). Otherwise, by (R3), $\omega^*(v) \geqslant d(v) - 4 - n_2(v) \times \frac{d(v)-4}{n_2(v)} = 0$. Since 4-vertex v does not participate in the discharging procedure, $\omega^*(v) = \omega(v) = d(v) - 4 = 0$.

Assume $d(v) = 3$. If v is rich, then it has non-negative charge. Suppose v is not rich. If v is incident to at least one 5^1-face, then $\omega^*(v) \geqslant 3 - 4 + 1 = 0$ by (R4.1). Otherwise let $N(v) = \{v_1, v_2, v_3\}$, denote by f_i the face that is incident to v such that vv_i and vv_{i+1} are its two boundary edges (indices modulo 3).

If v is adjacent to 2-vertex, say $d(v_1) = 2$, then by Claim 4.6, v_2 or v_3 has at least 4 big neighbours or in total they have at least 5 big neighbours. Since v is poor, w.l.o.g., assume $n_b(v_2) = 3$ and $n_b(v_3) = 2$. And the other neighbour of v_1, say v_1' has at least 4 big neighbours. Since $n_3(v_2) + n_b(v_2) \geqslant 4$, f_1 is incident to at most two poor vertices, thus $\tau(f_1 \to v) \geqslant \frac{1}{2}$ by (R4.2). If $d(v_3) = 3$, then f_3 is a 5^2-face, and $\tau(f_1 \to v) \geqslant \frac{1}{2}$ by (R4.2). Thus $\omega^*(v) \geqslant -1 + 2 \times \frac{1}{2} = 0$. Suppose $d(v_3) \geqslant 4$. Let v_2' and v_3' be the other two vertices of f_2. By Claim 4.5 and Claim 4.6, either both of them have degree at least 3 or one of them has degree 2 and the other has degree at least 4. Therefore, either $\tau(v_2 \to v) = \frac{1}{2}$ by (R1) or f_2 is a 5^2-face and $\tau(f_2 \to v) = \frac{1}{2}$, both imply that $\omega^*(v) \geqslant -1 + 2 \times \frac{1}{2} = 0$.

Suppose now v is not adjacent to any 2-vertices. If v is also not adjacent to any 3-vertex, then by the fact that v is poor, Claim 4.5 and Claim 4.6, for $i = 1, 2, 3$, f_i is either adjacent to three 3-vertices or it is a 5^2-face, therefore $\tau(f_i \to v) \geqslant \frac{1}{3}$ by (R4.2) and (R4.4).

If v is adjacent to exactly one 3-vertex, say v_1, then again f_2 is either incident to three 3-vertices or f_2 is a 5^2-face, therefore $\tau(f_2 \to v) \geqslant \frac{1}{3}$ by (R4.2) and (R4.4). Let $f_1 = vv_1x_1y_1v_2$ and $f_3 = vv_1x_2y_2v_3$. Since f_i, $i = 1, 2, 3$, cannot contain two 2-vertices, each of them sends at least $\frac{1}{6}$ to v by (R4.3.2) and (R4.4). First if v is a $3_{k,l}$-vertex, such that $k + l \geqslant 2$, then $\omega^*(v) \geqslant -1 + 2 \times \frac{1}{2} = 0$ by (R1) and (R4.2). For cases that $k + l \leqslant 1$, if $k = 1$, then we have $\omega^*(v) \geqslant -1 + \frac{1}{2} + \frac{1}{3} + 2 \times \frac{1}{6} = \frac{1}{6}$ by (R1). If $l = 1$, we consider following cases. If v is incident to a 5^3-face which contains no 2-vertex, then $\omega^*(v) \geqslant -1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 0$ by (R4.2) and (R4.3.2). Therefore we could always assume that $\tau(f_1 \to v) \geqslant \frac{1}{4}$ and $\tau(f_3 \to v) \geqslant \frac{1}{4}$ by (R4), which implies that $\omega^*(v) \geqslant -1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0$ by (R4.2). Thus $k = l = 0$, we suppose f_1, f_3 are 5^{3^+}-faces and f_2 is a 5^3-face that contains no 2-vertex. By (R4), we still have that $\tau(f_1 \to v) \geqslant \frac{1}{4}$ and $\tau(f_3 \to v) \geqslant \frac{1}{4}$.

1. First suppose both f_1 and f_3 do not contain any 2-vertex, if they are both 5^3-faces, then both of them send a charge of $\frac{1}{3}$ to v by (R4.4) and thus $\omega^*(v) \geqslant -1+3\times\frac{1}{3} = 0$. So let f_1 be a 5^4-face, then by Claim 4.13, $n_b(x_2) = 3$. Then when $d(y_2) = 3$, $\tau(f_3 \to v) \geqslant \frac{2}{3}$ by (R4.4), and when $d(y_2) \geqslant 4$, $\tau(f_3 \to v) = \frac{1}{2}$ by (R4.2). Thus $\omega^*(v) \geqslant -1 + \frac{1}{3} + \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$.

2. Either $d(x_1) = 2$ or $d(x_2) = 2$, by symmetry, we assume $d(x_1) = 2$. Then by Claim 4.6, x_2 has at least three big neighbours and y_1 has at least four big neighbours. If $d(y_2) = 3$, then by (R4.4), $\tau(f_3 \to v) \geqslant \frac{2}{3}$, and thus $\omega^*(v) \geqslant -1 + \frac{1}{3} + \frac{2}{3} + \frac{1}{3} = \frac{1}{3}$ by (R4.3.2). Assume now that $d(y_2) = 2$. If $n_3(x_2) \geqslant 2$ or $n_b(x_2) \geqslant 4$, then by (R4.3.2) and (R4.2), each of f_1 and f_3 will send v at least $\frac{1}{3}$, thus $\omega^*(v) \geqslant -1 + 3 \times \frac{1}{3} = 0$. So suppose $n_3(x_2) = 1$ and $n_b(x_2) = 3$. Then by Claim 4.4, $n_b(v_3) \geqslant 2$ and f_3 is a 5^2-face, which contradicts with our assumption that f_3 is a 5^{3^+}-face.

3. Either $d(y_1) = 2$ or $d(y_2) = 2$, by symmetry, we assume $d(y_1) = 2$. Then $d(x_1) \geqslant 4$. And we know that $d(x_2) \geqslant 3$. By Claim 4.4 and the fact that f_1 is a 5^{3^+}-face, $n_b(x_1) = 3$ or $n_b(v_2) = 3$. First suppose $n_b(v_2) = 3$, then by (R1), $\tau(v_2 \to v) \geqslant \frac{1}{2}$ and thus $\omega^*(v) \geqslant -1 + \frac{1}{2} + 2 \times \frac{1}{4} + \frac{1}{3} = \frac{1}{3}$. Suppose now that $n_b(x_1) = 3$, then $n_b(v_2) = 1$ since otherwise y_1 is rich. By Claim 4.4, $n_3(x_1) \geqslant 2$. By (R4.3.2), $\tau(f_1 \to v) \geqslant \frac{1}{3}$. If f_3 is also a 5^3-face, then we have $\tau(f_3 \to v) \geqslant \frac{1}{3}$. Otherwise f_3 is a 5^4-face such that all the small vertices have degree 3, then by Claim 4.13, the third neighbour of x_2 has at least three big neighbours. The third face of v_1 is either a 5^2-face or a 5^3-face with no 2-vertex, therefore, either v_1 is a $3_{1,1}$-vertex or

both v_1 and x_2 are $3_{1,0}$-vertices, by (R4.4), we always have $\tau(f_3 \to v) \geqslant \frac{1}{3}$. And $\omega^*(v) \geqslant -1 + 3 \times \frac{1}{3} = 0$.

Suppose v is adjacent to two 3-vertices, say v_2 and v_3. Let the other two vertices of f_i be x_i and y_i in the clockwise order, $i = 1, 2, 3$. By Claim 4.10, $n_b(v_1) \geqslant 2$. Again, since f_i, $i = 1, 2, 3$, cannot contain two 2-vertices, each of them sends at least $\frac{1}{6}$ to v by (R4.3.2) and (R4.4). Similarly if v is a $3_{k,l}$-vertex, such that $k + l \geqslant 2$ or $k = 1$, then $\omega^*(v) \geqslant 0$. Suppose $k = 0$ and $l = 1$, if v is incident to a 5^3-face which contains no 2-vertex, then $\omega^*(v) \geqslant -1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 0$ by (R4.2) and (R4.3.2). By (R4.3.2), f_i sends charge at least $\frac{1}{4}$ to v, except f_2 is a 5^5-face.

If one of x_2 and y_2 has degree 2, then by Claim 4.6, f_2 is a 5^2-face. Then $\omega^*(v) \geqslant -1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0$ by (R4.2). If one of x_2 and y_2 has degree 3, say $d(x_2) = 3$, then by Claim 4.13, either one of v_1 or y_1 has at least four big neighbours, or both of them have at least three big neighbours. Therefore either f_1 is a 5^2-face, or $n_b(v_1) \geqslant 3$ and $n_3(v_1) \geqslant 2$, both imply that $\omega^*(v) \geqslant -1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0$. Therefore, we may assume both x_2 and y_2 have degree at least four, and $\tau(f_2 \to v) \geqslant \frac{1}{3}$ by (R4.4).

If either y_1 or x_3 has degree at most 3, by symmetry say $d(y_1) \leqslant 3$, then by Claim 4.6 and Claim 4.13, either f_2 is a 5^2-face which implies $\omega^*(v) \geqslant -1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0$ by (R4.2), or both v_1 and x_2 has at least three big neighbours. If $n_3(v_1) \geqslant 2$, then $\tau(v_1 \to v) \geqslant \frac{1}{2}$ by (R1) and thus $\omega^*(v) > 0$. So we suppose $n_3(v_1) = 1$ and $n_b(v_1) = 3$. Then $d(x_1) \geqslant 4$ and $\tau(f_1 \to v) \geqslant \frac{1}{3}$. If $d(x_3) \leqslant 3$, then similarly either f_2 is 5^2-face or $\tau(f_3 \to v) \geqslant \frac{1}{3}$, we have $\omega^*(v) \geqslant 0$. Therefore, we assume $d(x_3) \geqslant 4$. If $d(y_3) \geqslant 4$, then f_3 is a 5^2-face which gives v enough charge. Otherwise $d(y_3) = 2$ since $n_3(v_1) = 1$, then by Claim 4.4, either f_3 is again a 5^2-face or f_3 is a 5^3-face and v_3 is incident to a 5^2-face and a 5^3-face which contains no 2-vertex, in both cases $\tau(f_3 \to v) \geqslant \frac{1}{3}$ by (R4.2) and (R4.3.2). So we always have $\omega^*(v) \geqslant -1 + 3 \times \frac{1}{3} = 0$.

In the following we may assume both y_1 and x_3 have degree at least 4. If both x_1 and y_3 have degree at least 3, then each f_i sends at least $\frac{1}{3}$ to v by (R4.2) or (R4.4), and $\omega^*(v) \geqslant -1 + 3 \times \frac{1}{3} = 0$. Assume $d(x_1) = 2$, if f_1 is a 5^2-face, then $\tau(f_1 \to v) \geqslant \frac{1}{2}$ and $\omega^*(v) \geqslant -1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{1}{12}$. Suppose f_1 is a 5^3-face. If v_2 is incident to a 5^2-face, since v_2 is incident to another 5^3-face which contains no 2-vertex, by (R4.3.2), $\tau(f_1 \to v) \geqslant \frac{1}{3}$. Otherwise we may assume the face $f' = v_2x_2x'_2y'_1y_1$ that v_2 incident is a 5^3-face, both x'_2 and y'_1 must have degree at most 3. We first derive that $d(y'_1) = 3$, since otherwise by Claim 4.5 and Claim 4.6, y_1 has at least four big neighbours. By Claim 4.4 and the fact that both x_1 and v are poor, $n_b(y_1) = 3$ and thus $\tau(f_1 \to v) \geqslant \frac{1}{3}$ by (R4.3.2). By symmetry, either $d(y_3) = 2$ or $d(y_3) \geqslant 3$, $\tau(f_3 \to v) \geqslant \frac{1}{3}$. Thus we have $\omega^*(v) \geqslant -1 + 3 \times \frac{1}{3} = 0$.

Finally suppose v is adjacent to three 3-vertices. Then by Claim 4.10, for $i = 1, 2, 3$, each f_i is a 5^3-face that contains no 2-vertex, thus sends at least $\frac{1}{3}$ to v by (R4.4), and $\omega^*(v) \geqslant -1 + 3 \times \frac{1}{3}$.

Assume now $d(v) = 2$ and let $N(v) = \{v_1, v_2\}$. If v is rich, or it is incident or adjacent to at least two of the following, then it would end up with a non-negative charge by (R1) and (R4.1): 5^+-vertex which has at least four 3^+-neighbours, and face with only one 3^--vertex, and 5^1-face.

Otherwise first suppose that v is adjacent to a 2-vertex v_1. Then by Claim 4.5, v_2 has at least five big neighbours and both incident faces are 5^2-faces and not special. So we have $\omega^*(v) \geqslant 2-4+1+2\times\frac{1}{2}=0$ by (R1) and (R4.2). Suppose now v is adjacent to a 3-vertex v_1. By Claim 4.6, v_2 has at least four big neighbours and both incident faces are 5^{3^-}-faces and not special, thus by (R1), (R4.2) and (R4.3), $\omega^*(v) \geqslant 2-4+1+2\times\frac{1}{2}=0$.

Suppose $d(v_1)=d(v_2)=4$. Then by the Fact 4.1, all the neighbours of v_1 and v_2 except v are big vertices. Thus the incident faces of v only contains one small vertex and v is rich.

Suppose $d(v_1)\geqslant 5$ and $d(v_2)=4$. If v_1 has at least four big neighbours, then by definition the incident faces of v are not special since otherwise v_2 is a special vertex, thus $\omega^*(v) \geqslant 2-4+1+2\times\frac{1}{2}=0$ by (R1) and Fact 4.2. Otherwise since $d(v_2)=4$, by Fact 4.1, $n_b(v_1)=3$. If $n_b(v_2)\leqslant 2$ or $n_3(v_1)\geqslant 1$, then $n_b(v_1)+n_3(v_1)\geqslant 4$, which implies that $\tau(v_1\to v)=1$ by (R1) and the incident faces of v are not special, so $\omega^*(v)\geqslant 2-4+1+2\times\frac{1}{2}=0$. Suppose now $n_b(v_2)=3$ and $n_3(v_1)=0$. If the incident faces of v are not special, then both of them contains only one small vertex which implies that v is rich. Otherwise by (R5) v would get enough charge such that $\omega^*(v)\geqslant 0$.

Suppose both v_1 and v_2 have degree at least 5. If one of the incident faces of v is special, then, by (R5), $\omega^*(v)\geqslant 0$. Otherwise, if at least one of v_1 and v_2 has at least four 3^+-neighbours, then $\omega^*(v)\geqslant 2-4+1+2\times\frac{1}{2}=0$ by (R1) and Fact 4.2. Suppose now both v_1 and v_2 have at most three 3^+-neighbours. Then by Fact 4.1, $n_b(v_1)=n_b(v_2)=3$, thus for $i=1,2$ $\tau(v_i\to v)\geqslant\frac{1}{2}$ by (R3). Since the incident faces are not special, each of them sends a charge of at least $\frac{1}{2}$ to v. Therefore $\omega^*(v)\geqslant 2-4+4\times\frac{1}{2}=0$.

Let $f\in F(G)$ and $d(f)=5$. If f is special, then by (R5) $\omega^*(f)\geqslant 5-4-1=0$. Thus we may assume f is not special. If f is incident to at most one small vertex, then by (R2) $\omega^*(f)\geqslant 5-4-1=0$. If f is a 5^1-face, then $\omega^*(f)\geqslant 5-4-1=0$ by (R3). If f is a 5^2-face. Then $\omega^*(f)\geqslant 5-4-2\times\frac{1}{2}=0$ by (R4.2). Suppose f is a 5^3-face. If $n_2(f)=2$ and $n_3(f)=1$, then by (R4.3.1) $\omega^*(f)\geqslant 5-4-2\times\frac{1}{2}=0$. If $n_2(f)=1$ and $n_3(f)=2$, then by (R4.3.2), either $\omega^*(f)\geqslant 5-4-\frac{1}{2}+2\times\frac{1}{4}=0$ or $\omega^*(f)\geqslant 5-4-\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=0$. Finally suppose f is a 5^{4^+}-face, it has non-negative charge by (R4.4).

It remains to show that every charge pot has non-negative charge. Observe that in a special face, every 4^+-vertex except the special vertex has at least 3 big neighbours by Fact 4.1. Let P be a charge pot with special vertex v which is obtained by k consecutive special faces $f_1, f_2, \ldots, f_k$. Let $v_1, v_2, \ldots, v_{k+1}$ be the consecutive 2-vertices on the special faces. Then by (R5) $\omega(P)=k$ and there are $k+1$ 2-vertices which will take charge from P. By (R3), v in total sends charge at least $(k+1)\times\frac{k+1+3-4}{k+1}=k$ to these 2-vertices. Let $N(v_1)=\{v,v_1'\}$, f_0 and f_1 be the incident faces of v_1. If $d(v_1')=4$, then f_0 contains only one small vertex and thus $\tau(f_0\to v)=1$ by (R2). Suppose $v_1'\geqslant 5$, then $\tau(v_1'\to v_1)\geqslant\frac{1}{2}$ by (R3). If f_0 is not special with respect to v_1', then $\tau(f_0\to v)\geqslant\frac{1}{2}$ by (R4.2). Otherwise by (R5) v_1 gets charge 1 from v_1' and the charge pot respect to v_1'. By symmetry v_{k+1} gets charge at least 1 which is not from v or the charge pot respect to v. Thus $\omega^*(P)\geqslant k-(2(k+1)-k-2)=0$. This completes our proof.

4.4 Concluding remarks

In this Chapter, using the result of Chapter 3 which itself is based on the 4-color theorem, we showed for every triangle free planar simple graph G, the signed graph (G, σ) has a packing number at least 5. Unlike the result of Chapter 3, the discharging technique used here is based on a planar embedding of G and thus cannot be applied to the class of K_5-minor-free graphs directly. However, an extension from planar graphs to K_5-minor-free graphs is already shown in [35].

It is unclear if this result can be proved independent of 4-color theorem. It is also not clear how important is the choice of the all negative signature. More precisely we would like to ask:

Question 4.1. What is the best possible lower bound on the packing number of planar signed graph of girth at least g?

Chapter 5

Separating signatures in signed planar graphs

This chapter is based on the following paper:

[42] R. Naserasr and W. Yu. Separating signatures in planar signed graphs. Accepted for publication in Discrete Appl. Math., 2023.

In this chapter, as a generalization of the packing number, instead of considering one signature and its equivalent signatures, we consider the following: given k signatures $\sigma_1, \sigma_2, \ldots, \sigma_k$ on a given graph G we say they are *separable* if there are signatures $\sigma_1', \sigma_2', \ldots, \sigma_k'$, where σ_i' is a switching of σ_i, such that the sets $E^-_{\sigma_i'}$ are pairwise disjoint. In particular, if we choose these k signatures to be σ, then being separable implies $\rho(G, \sigma) \geqslant k$. Given a graph G, if any set of k signatures on G are separable, then we say G has *k-separation property*.

The problem of packing number at least 2 is strongly connected to a notion of proper coloring of signed graphs first introduced by Zaslavsky in [50]. Recall that it is a coloring c of vertices of (G, σ) where colors are nonzero integers such that $c(x) \neq \sigma(xy)c(y)$. In a further study of this concept, Máčajová, Raspaud and Škoviera [32] conjectured that colors $\{\pm 1, \pm 2\}$ are enough for proper coloring of any signed planar simple graph. This conjecture was recently disproved by Kardoš and Narboni [28].

Connecting this two notions, it is shown in [39], a signed graph (G, σ) has packing number 2 if and only if $(G, -\sigma)$ admits a $\{\pm 1, \pm 2\}$-coloring, where $(G, -\sigma)$ is obtained from (G, σ) by turning the positive (resp. negative) edges to be negative (resp. positive). This implies that there exists a signed planar simple graph whose packing number is 1, see Chapter 3 for more details. In this chapter, we investigate sufficient conditions for a planar graph to have 2- or 3-separation property. We prove the followings.

Theorem 5.1. *Given integer i, $i \in \{3, 4, 5, 6\}$, any planar graph without a cycle of length i has 2-separation property.*

Theorem 5.2. *Every planar graph of girth at least 6 has 3-separation property.*

The last theorem is a corollary of a more general result on graphs of maximum average degree less than 3. In Section 5.1, we prove Theorem 5.1. Proof of Theorem 5.2 is provided in Section 5.2. In the last section, Section 5.3, we have concluding remarks where we mention connection to homomorphisms.

5.1 Separating 2 signatures in subclasses of signed planar graphs

In the rest of this section G will be a minimum counterexample to Theorem 5.1. We will see soon that this minimum counterexample has to be 2-connected and be of minimum degree at least 4. Thus in developing the terminology that is followed we consider G to be 2-connected and of minimum degree at least 4.

The counterexample G will be regarded as a plane graph that is a graph together with a planar embedding. As we consider 2-connected graphs every face is bounded by a cycle of G. We say that two faces (or cycles) are *adjacent* or *intersecting* if they share a common edge or a common vertex, respectively. Suppose that v is a k-vertex, and let $v_1, \dots, v_k$ be the neighbours of v in the clockwise order. For $i = 1, \dots, k$, $f_i(v)$ denotes the face incident with the vertex v with vv_i, vv_{i+1} (where the summation in the indices are taken modulo k) as boundary edges. As G is a plane graph of minimum degree at least 4, this is well defined.

For $a \in F(G)$, we write $a = [u_1u_2 \cdots u_l]$ if $u_1, u_2, \dots, u_l$ are the incident vertices of a in a cyclic order. As G is 2-connected and minimum degree at least four, each edge $e = u_ju_{j+1}$ of a face a determines a face adjacent to a at e. This face will be denoted by $f_j(a)$, where $j = 1, \dots, l$ and the summation in the indices are taken modulo l.

For two signatures σ and π on G, and for an edge $uv \in E(G)$, let $s_{\sigma\pi}(uv) = \{\sigma(uv)\pi(uv)\} \subseteq \{+,-\} \times \{+,-\}$. Observe that to separate σ and π is to find signatures σ', switching equivalent to σ, and π', switching equivalent to π, such that $s_{\sigma'\pi'}(uv) \neq --$ for every edge uv. For a vertex u define $S_{\sigma\pi}(u)$ as multiset $S_{\sigma\pi}(u) = [s_{\sigma\pi}(e) \mid e \text{ is incident with } u]$. Thus the order of $S_{\sigma\pi}(u)$ is the degree of u. Let $S^* = \{++, +-, -+\}$. We say a vertex v is *saturated* by σ and π if $S^* \subseteq S_{\sigma\pi}(v)$.

A path in G all whose vertices are of degree 4 in G is called a *light path*. Two paths are said to be *vertex disjoint* if their internal vertices are distinct. We say an m-face $a = [v_1v_2 \cdots v_m]$ is a *light face* if $d(v_i) = 4$ for all $i = 1, \dots, m$. A 5-face with four vertices of degree 4 and one vertex of degree 5 is called a *weak 5-face*. A weak 5-face is said to be *super weak 5-face* if it is adjacent to at least four triangles. For $x \in V(G) \cup F(G)$, let $n_3(x)$ denote the number of triangles incident or adjacent to x and $n_w(x)$ be the number of incident or adjacent weak faces.

It is well-known that every planar graph is 5-degenerate and that every triangle-free planar graph is 3-degenerate. It is shown in [49] that every planar graph without a 5-cycle is 3-degenerate. Similarly it is shown in [19] that every planar graph without 6-cycles is 3-degenerate. In the following, we will see that in a minimum counterexample to Theorem 5.1, the minimum degree is at least 4, which cannot be the case for 3-degenerate graphs. This would imply the claim of the theorem for each of the conditions of being triangle-free, having no 5-cycle or having no 6-cycle. What remains to prove is that if G is a planar graph with no 4-cycle, then any two signatures on it can be separated.

5.1.1 Structural properties of a minimum counterexample

Recall that G is a minimum counterexample to Theorem 5.1. That is to say either G has no triangle, or no 4-cycle, or no 5-cycle or no 6-cycle and there are signatures σ and π on G such that no matter how we switch them there is an edge which is assigned a negative sign by each of the two signatures.

The first observation is that G is connected, as otherwise separating signatures on each connected component, which would be possible by minimality, would be also a separation of the two signatures on the whole graph. Almost the same argument implies the following stronger claim.

Lemma 5.1. *The minimum counterexample G is 2-vertex-connected.*

Proof. Suppose to the contrary that v is a cut vertex of G. Let $G = G_1 \cup G_2$ such that v is the unique common vertex of G_1 and G_2, and there does not exist any edges between $V(G_1) - v$ and $V(G_2) - v$. Given two signatures σ and π on G, we consider subgraphs (G_1, σ), (G_1, π), (G_2, σ), and (G_2, π). By the assumption of the minimality of G, there are switchings σ_1 and π_1 on G_1 (resp. σ_2 and π_2 on G_2) of σ and π, respectively, such that they have no common negative edge.

In particular, in G_1 (resp. G_2), in order to get the switchings σ_1 and π_1 (resp. σ_2 and π_2) of σ and π, we could choose to switch at a subset V_1 (resp. V_2) of $V(G_1)$ (resp. $V(G_2)$) which does not contain v. Thus in G, if we switch at subset $V_1 \cup V_2$ which does not contain v as well, we find switchings σ' and π' of σ and π, such that σ' and π' have no common negative edge. This shows that a minimal counterexample cannot have a vertex cut of one vertex. $\square$

Lemma 5.2. *Given an edge $uv \in E(G)$ let $G' = G - uv$ and assume σ' and π' are switchings of σ and π, respectively, such that (G', σ') and (G', π') are separated. Then both u and v are saturated by σ' and π' in G'.*

Proof. Towards a contradiction and without loss of generality, assume $S^* \subsetneq S_{\sigma'\pi'}(u)$. Since σ' and π' have no common negative edge as signatures on $G - uv$, and G is counterexample, considering the extension of these signatures to G we have $s_{\sigma'\pi'}(uv) = --$. Assume $\alpha\beta \notin S_{\sigma'\pi'}(u)$, $\alpha\beta \in S^*$. If $\alpha = +$, switch σ' at u; if $\beta = +$, switch π' at u. After this operation, we have signatures σ'' and π'' both on G which agree with σ' and π' (respectively) on every edge that is not incident to u. Thus, by the choice of σ' and π', no edge which is not incident to u is negative in both. But, furthermore, based on our switchings $\{--\} \notin S(u)$ and thus σ'' and π'' are switchings of σ and π that are separated, a contradiction. $\square$

Corollary 5.1. *The minimum degree of G is at least 4.*

Thus as mentioned above, the case when G has no triangle or no 5-cycle or no 6-cycle is settled because any such a planar graph must be 3-degenerate.

Lemma 5.3. *Let P be a light path of G, $e \in P$. Assume σ_e and π_e are switchings of σ and π, respectively, such that (G, σ_e) and (G, π_e) have only e as their common negative edge. Then given an edge e' of P, by switching σ_e on a set X of vertices of P and switching π_e on a set Y of the vertices of P, for some choices of X and Y, we have signatures $\sigma_{e'}$ and $\pi_{e'}$ where e' is the only common negative edge of $(G, \sigma_{e'})$ and $(G, \pi_{e'})$.*

Proof. Suppose $P = v_1v_2 \cdots v_k$ and $e = \{v_iv_{i+1}\}$, where $i \in \{1, 2, \ldots k-1\}$. By our assumption $s_{\sigma_e\pi_e}(v_iv_{i+1}) = \{--\}$. By Lemma 5.2, $S_{\sigma_e\pi_e}(v_i) = S_{\sigma_e\pi_e}(v_{i+1}) = S^*$. With the same idea as in the proof of Lemma 5.2, and assuming $i \geqslant 2$, we may apply switchings at the vertex v_i so that $v_{i-1}v_i$ is the only common negative edge of the resulting two signatures. Similarly, assuming $i \leqslant k-2$ we may apply switchings at the vertex v_{i+1} so that $v_{i+1}v_{i+2}$ is the only common negative edge of the resulting two signatures. Continuing this process, and noting that each time switchings are only done on one of v_j's, $j = 2, \ldots k-1$, we have the desired claim. □

Lemma 5.4. *There is no pair of vertices connected by three vertex disjoint light paths.*

Proof. Assume to the contrary that P_1, P_2, P_3 are three vertex disjoint light uv-paths and label them as follows: $P_1 = ux_1 \cdots x_iv$, $P_2 = uy_1 \cdots y_jv$, and $P_3 = uz_1 \cdots z_kv$, where $i, j, k \geqslant 0$, noting that $k = 0$ means $P_3 = uv$ and that, since G is a simple graph, only one of these values can be 0. Thus, without loss of generality, we may assume $i \geqslant j \geqslant 1$ and $k \geqslant 0$. Since G has no 4-cycle, we also conclude that $i \geqslant 2$. Moreover, we may choose P_1, P_2, P_3 to be shortest subject to being internally vertex disjoint. This implies, in particular, that for any pair of non-consecutive vertices on a path P_i $(i = 1, 2, 3)$, they are not adjacent in G. Recalling that all vertices of a light path are of degree 4 in G, let t, w be the neighbours of u, v which are not on any of P_1, P_2, or P_3, respectively. Let $G' = G - ux_1$. By the minimality of G, assume σ' and π' are switchings of σ and π, respectively, such that (G', σ') and (G', π') are separated. Thus when σ' and π' are viewed as signatures on G we have $s_{\sigma'\pi'}(ux_1) = \{--\}$ and both u and x_1 are saturated. Noting that k is allowed to be 0, we consider two cases depending on this.

First consider the case $k \geqslant 1$, as depicted in Figure 5.1. We may apply Lemma 5.3 to switch only at the internal vertices of P_1 to obtain signatures σ'' and π'' such that x_iv is the only edge with $s_{\sigma''\pi''}(x_iv) = --$. Therefore, considering signatures σ'' and π'', and by Lemma 5.2, the vertex v must be saturated. Recall that in the process of getting σ'' and π'' from σ' and π' we are considering only switchings at the internal vertices of P_1. Furthermore, since P_i's chosen to be shortest, no internal vertex of P_i is adjacent to v. That means, in particular, that the signs of the three edges y_jv, z_kv, wv each incident to v remain untouched when switching σ' to σ'' and π' to π''. We conclude that

$$\{s_{\sigma'\pi'}(y_jv), s_{\sigma'\pi'}(z_kv), s_{\sigma'\pi'}(wv)\} = \{++, +-, -+\}. \tag{5.1}$$

Next, restarting from signatures σ' and π' and applying Lemma 5.3 to the path $x_1uy_1 \cdots y_jv$ (that is the path obtained from P_2 by adding the edge x_1u at the start), and as before, we conclude that

$$\{s_{\sigma'\pi'}(x_iv), s_{\sigma'\pi'}(z_kv), s_{\sigma'\pi'}(wv)\} = \{++, +-, -+\}. \tag{5.2}$$

In this argument that $k \geqslant 1$ helps us to confirm that the signs of the three edges incident to v other than y_jv remain the same.

Equations 5.1 and 5.2 imply that $s_{\sigma'\pi'}(x_iv) = s_{\sigma'\pi'}(y_jv)$.

Similarly, considering paths P_1 and $x_1uz_1 \cdots z_kv$ we conclude that $s_{\sigma'\pi'}(x_iv) = s_{\sigma'\pi'}(z_kv)$. However, this leads to contradiction with either of the identities 5.1 and 5.2. This concludes the statement for the case that $k \geqslant 1$.

Now assume $k = 0$, that is to say uv is an edge of G, this case is depicted in Figure 5.1. First suppose that, except for the edge uv, no vertex of P_1 is connected to a vertex of P_2. Our first claim in this case is that $s_{\sigma'\pi'}(uy_1) = s_{\sigma'\pi'}(y_1y_2) = \cdots = s_{\sigma'\pi'}(y_jv)$. That is because by applying Lemma 5.3 and Lemma 5.2 to the path $x_1uy_1y_2\cdots y_jv$ we get that $S_{\sigma'\pi'}(y_l) - s_{\sigma'\pi'}(y_ly_{l-1}) = S^*$ and by applying the same lemma to the path $ux_1x_2\cdots x_ivy_jy_{j-1}\dots y_1$ we get that $S_{\sigma'\pi'}(y_l) - s_{\sigma'\pi'}(y_ly_{l+1}) = S^*$.

Next we claim that $s_{\sigma'\pi'}(x_iv) = s_{\sigma'\pi'}(uv)$. That is for similar reasons as the previous claim and by considering the two paths P_1 and x_1uv. Furthermore, applying Lemma 5.2 to signature σ'' and π'' which have only x_iv as common negative edge, and are obtained from switching of σ' and π' (respectively) on internal vertices of P_1, we conclude that:

$$\{s_{\sigma'\pi'}(uv), s_{\sigma'\pi'}(y_jv), s_{\sigma'\pi'}(wv)\} = \{++, +-, -+\}. \tag{5.3}$$

Recall that u is saturated by σ' and π' where ux_1 is negative in both signatures. This means

$$\{s_{\sigma'\pi'}(ut), s_{\sigma'\pi'}(uy_1), s_{\sigma'\pi'}(uv)\} = \{++, +-, -+\}. \tag{5.4}$$

Comparing identities 5.3 and 5.4 we have: $s_{\sigma'\pi'}(ut) = s_{\sigma'\pi'}(vw)$.

Observe that when applying Lemma 5.3 to get uy_1 as the only common negative edge, we apply switchings at u in one or both of the signatures. Assuming the new signatures are σ'' and π'' one observes that $s_{\sigma''\pi''}(ux_1) = s_{\sigma'\pi'}(uy_1)$ and thus $s_{\sigma''\pi''}(uv) = s_{\sigma'\pi'}(ut)$. Therefore, $s_{\sigma''\pi''}(uv) = s_{\sigma'\pi'}(vw)$.

If we now apply Lemma 5.3 to σ'' and π'' on the path P_2 so to have y_jv as the only common negative edge, as we will not change signs of the other three edges incident with v we will end up with a vertex v which is not saturated, contradicting Lemma 5.2.

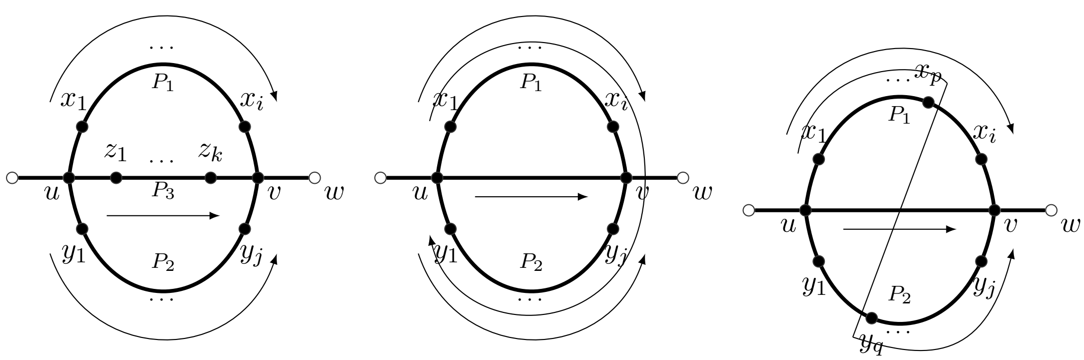

Figure 5.1: 3 disjoint light paths between u and v

For the final case, suppose beside uv, there exists another edge connecting a vertex of P_1 to a vertex of P_2. Let x_py_q be such an edge. Since $i \geqslant 2$, and by exchanging the roles of u and v, if needed, we may assume that $p \leqslant i-1$. In this case, as before we apply Lemma 5.3 to the following three paths: P_1, x_1uv, and $ux_1 \cdots x_py_q \cdots y_jv$. From the first we conclude that $\{s_{\sigma'\pi'}(uv), s_{\sigma'\pi'}(y_jv), s_{\sigma'\pi'}(wv)\} = \{++,+-,-+\}$. From the second we conclude that $\{s_{\sigma'\pi'}(x_iv), s_{\sigma'\pi'}(y_jv), s_{\sigma'\pi'}(wv)\} = \{++,+-,-+\}$. And the last one implies $\{s_{\sigma'\pi'}(uv), s_{\sigma'\pi'}(x_iv), s_{\sigma'\pi'}(wv)\} = \{++,+-,-+\}$. Comparing the first two we conclude that $s_{\sigma'\pi'}(uv) = s_{\sigma'\pi'}(x_iv)$, then first with second $s_{\sigma'\pi'}(uv) = s_{\sigma'\pi'}(y_jv)$ which contradicts, say, the third identity. □

Corollary 5.2. *There are no adjacent light faces in G.*

We may now apply discharging technique to conclude our claim.

5.1.2 Discharging for planar graphs without 4-cycles

In this section, we apply discharging technique to complete the proof of Theorem 5.1 for the case of C_4-free planar graphs.

We define a weight function ω on the vertices and faces of G by letting $\omega(v) = d(v)-4$ for each $v \in V(G)$ and $\omega(f) = d(f) - 4$ for $f \in F(G)$. It follows from Euler's formula and the relation $\sum_{v \in V(G)} d(v) = \sum_{f \in F(G)} d(f) = 2|E(G)|$ that the total sum of weights of the vertices and faces satisfies the following

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (d(f) - 4) = -8.$$

Next we design appropriate discharging rules and redistribute weights accordingly. Once the discharging is finished, a new weight function ω^* is produced. The total sum of weights is kept fixed when the discharging is in process. Nevertheless, after the discharging is complete, we will show that $\omega^*(x) \geqslant 0$ for all $x \in V(G) \cup F(G)$. This contradiction implies that no such counterexample exists.

Let v be vertex of degree 4 whose neighbours in clockwise orientation are v_1, v_2, v_3, and v_4. Let f_1, f_2, f_3, and f_4 be the face containing v_1vv_2, v_2vv_3, v_3vv_4, and v_4vv_1 respectively. If $d(v_3) = d(v_4) = 4$, $d(v_1) = d(v_2) \geqslant 5$, $d(f_2) = d(f_4) = 3$, $d(f_3) = 5$, and $d(f_1) \geqslant 5$, then we say f_3 is a *receiver* of f_1.

For $x, y \in V(G) \cup F(G)$, let $\tau(x \to y)$ denote the amount of weights transferred from x to y.

Our first discharging rule is as follows:

$R1:$ Each 5^+-face sends $\frac{1}{3}$ to each adjacent 3-face and $\frac{2}{15}$ to each of its receiver.

Let v be a 5-vertex with $f_1, f_2, \ldots, f_5$ being the faces incident to v. Assume f_1 and f_3 are triangles and, furthermore, that f_4 is a super weak 5-face. Then it is easily observed that f_5 is not a super weak 5-face.

The next two discharging rules are as follows:

$R2$: If $d(v) = 5$, $n_3(v) = 1$, say $d(f_1) = 3$, then let $\tau(v \to f_2) = \tau(v \to f_5) = \frac{1}{3}$.

$R3$: If $d(v) = 5$ and $n_3(v) = 2$, say $d(f_1) = d(f_3) = 3$, then $\tau(v \to f_2) = \frac{2}{3}$. Furthermore, if there exists one super weak 5-face f', $f' \neq f_2$, then $\tau(v \to f') = \frac{1}{3}$, otherwise $\tau(v \to f_4) = \tau(v \to f_5) = \frac{1}{6}$.

The remaining two rules are about 6^+-vertices.

$R4$: If $d(v) \geqslant 6$ and f is a face incident to v and adjacent to one triangle also incident to v, then $\tau(v \to f) = \frac{1}{3}$.

$R5$: If $d(v) \geqslant 6$ and f is a face incident to v and adjacent to two triangles each incident to v, then $\tau(v \to f) = \frac{2}{3}$.

In the following, we will show that $\omega^*(x) \geqslant 0$ for all $x \in V(G) \cup F(G)$.

First we consider vertices, let $v \in V(G)$. By Corollary 5.1, $d(v) \geqslant 4$. Note that no 4-vertex participates in discharging argument, so $\omega^*(v) = \omega(v) = d(v) - 4 = 0$ for any 4-vertex v. Next we consider 5-vertices. Let v be any such a vertex, then $\omega(v) = 1$. By the fact that G contains no 4-cycle we have $0 \leqslant n_3(v) \leqslant 2$. If $n_3(v) = 0$, then the charge of v is not changed, i.e., $\omega^*(v) = \omega(v) = 1$. If $n_3(v) = 1$, the charge of v is changed (only) by the $R2$, and in this case $\omega^*(v) = \omega(v) - 2 \times \frac{1}{3} = \frac{1}{3}$. If $n_3(v) = 2$, then $R3$ is the only rule that changes the charge of v and under this rule at most a charge of 1 is given from v to its incident face. Thus $\omega^*(v) \geqslant 0$.

It remains to consider 6^+-vertices. Let v be such a vertex. $d(v) \geqslant 6$. For $i = 1, 2$, let $m_i(v)$ denote the number of incident faces adjacent to i triangles each incident to v. Observe that, by definition, $m_1(v) + 2m_2(v) \leqslant 2n_3(v) \leqslant d(v)$ (the latter inequality because of being C_4-free). In applying $R3$ the vertex v loses a charge of $\frac{m_1(v)+2m_2(v)}{3}$. Thus $\omega^*(v) = d(v) - 4 - \frac{m_1(v)+2m_2(v)}{3}$. Therefore, $\omega^*(v) \geqslant d(v) - 4 - \frac{d(v)}{3}$. As $d(v) \geqslant 6$ we have $\omega^*(v) \geqslant 0$.

Now we consider faces, let $f \in F(G)$. First assume $d(f) = 3$, in other words f is a triangle. Recall that original charge $\omega(f) = -1$. Since G has no C_4, each of the faces adjacent to f is of size at least 5. Then by rule $R1$, each of them sends a charge of $\frac{1}{3}$ to f and thus $\omega^*(f) = 3 - 4 + 3 \times \frac{1}{3} = 0$.

Next we consider 5-faces, let $f = [v_1 \cdots v_5]$ be such a face. For the original charge of f we have $\omega(f) = 5 - 4 = 1$. If f is adjacent to at most two triangles, then f gives a charge of $\frac{1}{3}$ to each of the triangles it is adjacent to and it has at most one receiver, so can only lose a charge of $2 \times \frac{1}{3} + \frac{2}{15} = \frac{4}{5}$, thus the final charge is at least $\frac{1}{5}$.

Suppose f is adjacent to precisely 3 triangles. If f has no receiver, then it only loses charge by $R1$ and by this rule loses exactly a charge of $3 \times \frac{1}{3} = 1$, hence $\omega^*(f) = 0$. If f has exactly one receiver, let v_2 be the common vertex of f and its receiver. Then, by the definition of a receiver, v_1, v_3 each has degree at least 5. We now consider the position of the third triangle adjacent to f. If it is one of f_3 or f_5, say f_3, then by $R3$ or $R5$, depending on if $d(v_3) = 5$ or $d(v_3) \geqslant 6$, the vertex v_3 gives a charge of $\frac{2}{3}$ to f, concluding that $\omega^*(f) \geqslant \frac{8}{15}$. Otherwise f_4 is the third triangle adjacent to f. In such a case the two faces f_3 and f_5 are 5^+-faces. We claim that neither is a super weak 5-face. By contradiction, suppose f_3 is a super weak 5-face. Then, it must be adjacent to at least four triangles. As f is not a triangle, all the other faces adjacent to f_3 are triangles. This implies that vertices v_3 and v_4 are each of degree at least 5, but this contradicts the second condition of being a super negative 5-face which is to have four vertices of degree 4. If f_3 (or f_5) is a 6^+-face, then, by $R4$, it gives a charge of $\frac{1}{3}$ to f, raising $\omega^*(f)$ to at least $\frac{1}{5}$. If they are both 5-faces, then, by $R3$, each of v_1 and v_3 gives a charge of $\frac{1}{6}$ to f. The final charge of f in this is at least $1 + \frac{1}{6} - 3 \times \frac{1}{3} - \frac{2}{15} = \frac{1}{30}$.

Suppose f is adjacent to 4 triangles and by symmetry assume f_1, f_2, f_3, and f_4 are the triangles. The receiver face implies that there are at most two receivers for f, and moreover if there is at least one, then one of v_2, v_3 or v_4 has to be of degree at least 5. If one of v_2, v_3 or v_4 is of degree at least 5, either by $R3$ or by $R5$ it will give a charge of $\frac{2}{3}$ to f and thus the final charge of f would be at least $1 + \frac{2}{3} - 4 \times \frac{1}{3} - 2 \times \frac{2}{15} = \frac{1}{15}$. Let now assume the vertices v_2, v_3, and v_4 are all of degree 4. In such case, if v_5 and v_1 each has degree at least 5 or one of them has degree at least 6, then f_5 cannot be a weak face and either by applying $R3$ to both v_1 and v_5 or applying $R4$ to the one which is a 6^+-vertex, a total charge of at least $\frac{1}{3}$ is given to f and thus the final charge of f is non-negative. If one of v_1 and v_5 is degree 4 and the other, say v_5 is of degree 5, then f is a super weak 5-face and thus by $R3$ the vertex v_5 will give a charge of $\frac{1}{3}$ to f, resulting a final charge of f to be positive. If all vertices $v_1, \ldots, v_5$ are of degree 4, i.e., f is a light face, then since there is no adjacent light faces (Corollary 5.2) for each of the triangles $f_1, \ldots, f_4$ the vertex of f_i which is not on f is a 5^+-vertex. Then f is a receiver for the face adjacent to f_1 and f_2 and for the face adjacent to f_2 and f_3 and also for the face adjacent to f_3 and f_4. It, therefore, receives a charge of $\frac{2}{15}$ from each of these 3 for a final charge of $\omega^*(f) \geqslant 1 + 3 \times \frac{2}{15} - 4 \times \frac{1}{3} = \frac{1}{15}$.

Finally we consider the case where all faces adjacent to f are triangles. Recall that f has at most two receivers. So it loses at most $5 \times \frac{1}{3} + 2 \times \frac{2}{15}$. If two of v_i's are 5^+ vertices, then either by $R3$ or by $R5$ they each gives a charge of $\frac{2}{3}$ to f and the final charge of f is positive. If only one of v_i's, say v_1, is a 5^+-vertex, then f has no receiver and only loses a charge of $5 \times \frac{1}{3}$ but gains $\frac{2}{3}$ from v_1 and again the final charge would be non-negative. If none of v_i's is a 5^+-vertex, i.e., f is a light face, by Corollary 5.2, for each of the triangles $f_1, \ldots, f_5$ the vertex of f_i which is not on f is a 5^+-vertex. Thus f is a receiver of five faces determined by consecutive triangles around it. Hence by $R1$, it receives $5 \times \frac{2}{15}$ from each of these five faces, to have a final charge of 0. This conclude all the cases for a 5-face.

Next assume that $f = [v_1 \cdots v_6]$ is a 6-face. Then $\omega(f) = 2$. If f is adjacent to at most 5 triangles, then it has at most two receivers, and hence it loses at most $5 \times \frac{1}{3} + 2 \times \frac{2}{15}$ (all in $R1$) hence $\omega^*(f) \geqslant \frac{1}{15}$. If all the six faces adjacent to f are triangles, we consider two possibilities depending on the degrees of $v_1, \ldots, v_6$. If at least one of them is a 5^+-vertex, then either by $R3$ or by $R5$ it gives a charge of $\frac{2}{3}$ to f. As f can have at most three receivers, the final charge of f remains non-negative. If all vertices on f are of degree 4, then f has no receiver and the final charge of f is 0.

Finally we consider 7^+-faces. Recall that faces only lose charge by $R1$. There are at most $d(f)$ triangles adjacent to f, and it can have at most $\lceil \frac{d(f)}{2} \rceil$ receivers. Thus for the final charge of f we have $\omega^*(f) = d(f) - 4 - \frac{1}{3}d(f) - \frac{1}{2} \times \frac{2}{15}d(f) = \frac{3}{5}d(f) - 4 \geqslant \frac{1}{5}$. This completes the proof.

5.2 Separating 3 signatures in signed planar graphs of girth 6

In this section we provide a maximum average degree condition which is sufficient for any three signatures on a graph to be separated. Theorem 5.2 will be immediate consequence then.

Theorem 5.3. *Every simple graph of maximum average degree less than 3 has a 3-separation property.*

Proof. Let G be a minimum counterexample. That, in particular means there are three signatures σ_1, σ_2, and σ_3 on G that are not separable but for any edge e, the restrictions of the three signatures on $G - e$ are separable. After proving a few claims, and using discharging technique then we will show that G itself must have average degree at least 3 contradicting our hypothesis on the maximum average degree of G.

For three signatures σ_1, σ_2 and σ_3 on G, and for an edge $uv \in E(G)$, let $s_{\sigma_1\sigma_2\sigma_3}(uv) = \{\sigma_1(uv)\sigma_2(uv)\sigma_3(uv)\} \subseteq \{+,-\} \times \{+,-\} \times \{+,-\}$. For a vertex u define a multiset $S_{\sigma_1\sigma_2\sigma_3}(u) = [s_{\sigma_1\sigma_2\sigma_3}(e) | e \in E_u]$, where E_u is the set of edges incident to u. We may use $s(uv)$ and $S(u)$ when the signatures are clear from the context. Let $S^* = \{+++, -++, +-+, ++-\}$.

The first observation, which is easy to derive, is that G is 2-connected. Thus in particular the minimum degree is at least 2. To achieve our goal then we have three claims about the neighbourhood of vertices of degree 2.

Claim 1. Both neighbours of a 2-vertex v in G have degree at least 4.

Proof of the claim. Let $N(v) = \{v_1, v_2\}$ and assume to the contrary, that $d(v_2) \leqslant 3$. Let $G' = G - vv_2$. By the minimality of G, assume σ_1', σ_2', and σ_3' are switchings of σ_1, σ_2, and σ_3, respectively, such that (G', σ_1'), (G', σ_2'), and (G', σ_3') are separated. Since $d_{G'}(v) = 1$, and by a switching at v in any signature that needs, we may assume $s_{\sigma_1'\sigma_2'\sigma_3'}(vv_1) = \{+++\}$. When σ_1', σ_2', and σ_3' are viewed as signatures on G, vv_2 is the only edge not satisfying the condition which means at least two of the signatures must assign negative to vv_2. If one of them, say σ_3' assigns positive to vv_2, then by switching v at the signature σ_2' (or σ_1') we have separation. Thus we may assume $s_{\sigma_1'\sigma_2'\sigma_3'}(vv_2) = \{---\}$.

At this point, it suffices to find one or two signatures, σ_i' and σ_j', such that if σ_i' or both σ_i' and σ_j' are switched at v_2, then vv_2 is still the only edge not satisfying our condition. If we manage to find σ_i', or σ_i' and σ_j', then we may also switch signature σ_l', $l \notin \{i, j\}$, at v. After these switchings, vv_1 will be negative at one signature only, and vv_2 with be either positive in all or negative in just one signature, and thus we have three separated signatures. To choose σ_i' and possibly σ_j' among σ_1', σ_2', and σ_3' we consider the two edges, e_1 and e_2 incident to v_2 but different from vv_2. If they are both negative in a signature, we choose that one to be σ'_i and no need for a second. If each of the edges is assigned only positive sign by each of the signatures, then σ'_i can be any of the three signatures and again no need for a second choice. Otherwise, we note that at most two of the signatures can assign different signs to e_1 and e_2. If only one, then we choose that signature to be σ'_i and if two we take them both to be σ'_i and σ'_j. $\diamond$

Claim 2. A 4-vertex v can have at most two 2-neighbours.

Proof of the claim. Let $N(v) = \{v_1, v_2, v_3, v_4\}$. Toward a contradiction assume that $d(v_i) = 2$ for $i = 1, 2, 3$. For each i, $i = 1, 2, 3$, let the other neighbour of v_i be v'_i. Let $G' = G - vv_1$. By the minimality of G, we have signatures σ'_1, σ'_2, and σ'_3 as switchings of σ_1, σ_2, and σ_3, respectively, such that (G', σ'_1), (G', σ'_2), and (G', σ'_3) are separated. In what follows, we consider signatures σ'_1, σ'_2, and σ'_3 on G. Again since $d_{G'}(v_1) = 1$, without loss of generality, we may assume $s(v_1v'_1) = \{+++\}$. The same argument as in the previous case then implies that $s(vv_1) = \{---\}$.

If $S(v) \cap S^* \leqslant 2$, then we continue the same argument as in the previous case, where v is a neighbour of the 2-vertex v_1 and to our purpose it is of degree $|S(v) \cap S^*| + 1$. So we assume $S(v) \cap S^* = 3$. We observe that by switching at v_2, in the signatures that are needed, we may exchange $s(vv_2)$ and $s(v_2v'_2)$. If after such switchings the previous condition holds, we are done. If not, either $s(vv_2) = s(v_2v'_2)$ in which case by switchings at v_2 we may conclude that $s(vv_2) = s(vv_3)$ and then we are done as before, or $s(v_2v'_2)$ is distinct from each of $s(vv_2)$, $s(vv_3)$, and $s(vv_4)$. Repeating the same argument we conclude that $s(v_3v'_3) = s(v_2v'_2)$. We may now do enough switchings at v_2 and v_3 so that $s(vv_2) = s(vv_3)$. Then the process can be completed as before. $\diamond$

Claim 3. A 5-vertex v can have at most four 2-neighbours.

Proof of the claim. Let $N(v) = \{v_1, \dots, v_5\}$. Assume to the contrary that $d(v_i) = 2$ for $i = 1, \dots, 5$. We name the other neighbour of v_i as v'_i. Let $G' = G - vv_1$. By the minimality of G, assume σ'_1, σ'_2, and σ'_3 are switchings of σ_1, σ_2, and σ_3, respectively, such that (G', σ'_1), (G', σ'_2), and (G', σ'_3) are separated. As in the previous two cases, we may assume $s(v_1v'_1) = \{+++\}$ and $s(vv_1) = \{---\}$. Furthermore, by switching at v_i's, if necessary, we can assume that none of $s(vv_i)$, $i = 2, \dots, 5$, is $\{+++\}$. Thus for some i and j, $2 \leqslant i < j \leqslant 5$, we have $s(vv_i) = s(vv_j)$. At this point we note that in the proof of Claim 2 we never applied a switching at v_4. Thus we may now continue the same proof as in the Claim 2 by treating v_i and v_j as v_4 and not switching at these two vertices. $\diamond$

Finally to complete the proof we show that the three forbidden configurations of Claims 1, 2, and 3 imply an average degree of at least 3.

We first define ω on the vertices of G by letting $\omega(v) = d(v)$ for each $v \in V(G)$. The single discharging rule is as follows.

R' : Each 4^+-vertex sends $\frac{1}{2}$ to each 2-neighbour.

Let $\omega^*(v)$ be the charge of v after applying the rule. Let $v \in V(G)$. As observed before, $d(v) \geqslant 2$. If $d(v) = 2$, then by Claim 1, v is adjacent to two vertices of degree at least 4. Thus, $\omega^*(v) = 2 + 2 \times \frac{1}{2} = 3$ by (R'). The discharging rule does not change $\omega(v)$ if $d(v) = 3$. If $d(v) = 4$, then by Claim 2, v has at most two 2-neighbours, thus $\omega^*(v) \geqslant 4 - 2 \times \frac{1}{2} = 3$. When $d(v) = 5$, by Claim 3, v has at most four 2-neighbours, thus $\omega^*(v) \geqslant 5 - 4 \times \frac{1}{2} = 3$. Finally if $d(v) \geqslant 6$, then $\omega^*(v) \geqslant d(v) - \frac{d(v)}{2} = \frac{d(v)}{2} \geqslant 3$. □

5.3 Conclusion

We have known that problem of packing signatures in signed graphs relates to some of the most prominent problems in graph theory such as the four-color theorem and edge-coloring problems as shown in Chapter 3. The question of separating a given set of k signatures captures the k-packing of signature problem because one can simply take k identical signatures. The question then can be translated back to a homomorphism problem as follows.

A multi-signed graph, denoted $(G, \sigma_1, \sigma_2, \dots, \sigma_l)$, is a graph G together with l signatures. A multi-signed graph $(G, \sigma_1, \sigma_2, \dots, \sigma_l)$ is said to admit a homomorphism to a multi-signed graph $(H, \pi_1, \pi_2, \dots, \pi_l)$ if there is a mapping f of vertices and edges of G to vertices and edges of H, respectively, which is a homomorphism of (G, σ_i) to (H, π_i) for every i, $i = 1, 2, \dots, l$. That is to say incidences and adjacencies are preserved, and the sign of any closed walk in (G, σ_i) is the same as the sign of its image in (H, π_i).

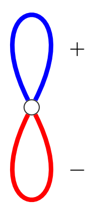

Figure 5.2: (L_1, σ)

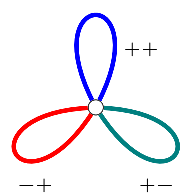

Figure 5.3: $(L_2, \sigma_1, \sigma_2)$

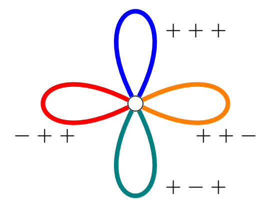

Figure 5.4: $(L_3, \sigma_1, \sigma_2, \sigma_3)$

Given an integer l, let L_l be the multi-signed graph on a single vertex with $l + 1$ loops $e_0, e_1, \dots e_l$ where e_0 is assigned a positive sign by each of the signatures and e_i is assigned a negative sign by σ_i and positive sign by all other signatures. The cases $l = 1, 2, 3$ are presented in Figures 5.2, 5.3, and 5.4. It is then immediate to restate the separating problem we have studied here as a homomorphism problem.

Theorem 5.4. *A multi-signed graph $(G, \sigma_1, \sigma_2, \dots, \sigma_l)$ admits a separation if and only it admits a homomorphism to L_l.*

Part III

Vertex decomposition of sparse graphs

Chapter 6

An (F_3, F_5)-partition of planar graphs of girth at least 5

This chapter is based on the following paper:

[9] M. Chen, A. Raspaud, W. Wang, and W. Yu. An (F_3, F_5)-partition of planar graphs with girth at least 5. Discrete Math., 346(2):Paper No. 113216, 17, 2023.

Let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ denote k classes of graphs. Recall that a graph G admits a $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$-partition, if $V(G)$ can be partitioned into k vertex subsets $V_1, \ldots, V_k$ such that the subgraph $G[V_i]$ belongs to $\mathcal{C}_i$ for each $1 \leqslant i \leqslant k$. We use F, F_d, Δ_d and I to denote the class of forests, the class of forests with maximum degree at most d, the class of graphs with maximum degree at most d, and the class of empty graphs, respectively. Obviously, $I = \Delta_0 = F_0$ and $\Delta_1 = F_1$. The famous 4-color theorem guarantees that every planar graph admits an (I, I, I, I)-partition.

In this chapter, we study vertex partitions of graphs under restriction on girth condition. Recall that $\mathcal{PG}_g$ denote the family of planar graphs of girth at least g. It has been proved in [33] that there is a graph belonging to $\mathcal{PG}_4$ having no $(\Delta_{d_1}, \Delta_{d_2})$-partition for any non-negative integers d_1 and d_2. Therefore, we are aiming to find a refinement of forest partitions of $\mathcal{PG}_5$. More specifically, we prove the following.

Theorem 6.1. *Every graph in $\mathcal{PG}_5$ admits an (F_3, F_5)-partition.*

This is an improvement of a result in [12] stating that every graph in $\mathcal{PG}_5$ admits a (Δ_3, Δ_5)-partition. Our proof is based on the discharging technique. Assume that G is a minimum counterexample to Theorem 6.1. In Section 6.1, we first give some basic notations. In Section 6.2, we study the structural properties of the minimum counterexample. Finally in Section 6.3, the discharging technique is employed to prove that G does not exist, which finishes the proof.

6.1 Preliminaries

Arguing by contradiction, we assume that $G = (V, E)$ is a counterexample to Theorem 6.1 minimizing $|V(G)|$. Embedding G into the plane, we get a plane graph $G = (V, E, F)$ with the face set F. It is obvious that G is connected with $g(G) \geqslant 5$. If G contains a 1-vertex, say v, then let u be the unique neighbour of v. Take an (F_3, F_5)-partition of $G - v$. One may easily reach an (F_3, F_5)-partition of G by adding v to F_i, where $i \in \{3, 5\}$ and $u \notin F_i$. Thus, in what follows, assume that G is a connected graph of minimum degree at least 2.

For $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f and write $f = [u_1u_2 \dots u_n]$ if $u_1, u_2, \dots, u_n$ are the vertices of $b(f)$ appearing in a boundary walk of f. For $x \in V(G) \cup F(G)$, we use $n_i(x)$ to denote the number of i-vertices adjacent or incident to x. Let $uv \in E(G)$. If $d(v) = k$, then we call v a *k-neighbour* of u. We may similarly define a k^+-neighbour or a k^--neighbour of u. Let $i \in \{3, 5\}$. Given a (partial) (F_3, F_5)-partition of $G' \subsetneq G$, a vertex v is said to be an *F_i-vertex* if $v \in F_i$. An *F_i-neighbour* of v is an F_i-vertex adjacent to v. Furthermore, we call v *F_i-saturated* if v is an F_i-vertex with exactly i F_i-neighbours. By definition, it is easy to see that an F_i-saturated vertex has at least i neighbours.

Let f be a 5-face of G. We call f *good*, *weak*, and *bad* if $n_2(f) = 0$, $n_2(f) = 1$ and $n_2(f) = 2$, respectively, as shown in Figure 6.1. Let v be a 3-vertex in G. We call v *heavy* if $n_{5^+}(v) \geqslant 2$, and *light* otherwise, as shown in Figure 6.2. For our convenience, we use $n_{3^h}(u)$ and $n_{3^l}(u)$ to denote the number of heavy 3-vertices and light 3-vertices adjacent to u, respectively.

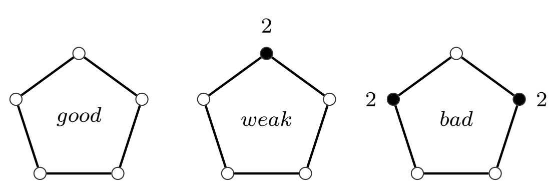

Figure 6.1: good, weak and bad 5-faces

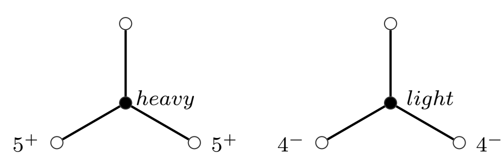

Figure 6.2: heavy and light 3-vertices

For all figures in this part, a vertex is represented by a solid node when all of its incident edges are drawn; otherwise it is represented by a hollow node.

6.2 Structural analysis of minimum counterexample

6.2.1 Elementary structural lemmas

Lemma 6.1. [12] *Every* 2*-vertex is adjacent to a* 5^+*-vertex and a* 7^+*-vertex.*

Lemma 6.2. [12] *No* 3*-vertex can be incident to any bad* 5*-face.*

Lemma 6.3. *Let v be a* 2*-vertex adjacent to v_1 and v_2 such that $d(v_1) \leqslant 6$. Then v_1 is F_3-saturated and v_2 is F_5-saturated in $G - v$.*

Proof. By Lemma 6.1, $d(v_2) \geqslant 7$. Clearly, $G - v$ admits an (F_3, F_5)-partition due to the minimality of G. If $v_1, v_2 \in F_i$ for some fixed $i \in \{3, 5\}$, then it is easy to get an (F_3, F_5)-partition of G by putting v to F_j such that $j \in \{3, 5\} \backslash \{i\}$. Otherwise, we deduce that v_1 and v_2 belong to different forest partitions. If $v_1 \in F_5$, then $v_2 \in F_3$, and thus we can directly add v to F_5, and further move v_1 to F_3 if v_1 is F_5-saturated in $G - v$. So now assume that $v_1 \in F_3$ and $v_2 \in F_5$. If one fails to put v to F_3 or F_5, then we obtain immediately that v_1 is F_3-saturated and v_2 is F_5-saturated in $G - v$. □

Lemma 6.4. *Let v be a* 2*-vertex adjacent to v_1 and v_2 such that $d(v_1) = 5$ and $d(v_2) = 7$. Let $N_G(v_1) = \{v, x_1, \ldots, x_4\}$ and $N_G(v_2) = \{v, y_1, \ldots, y_6\}$. Then the following holds:*
(1) If $n_{4^+}(v_1) = 1$, say $d(x_1) \geqslant 4$, then $x_1 \in F_5$;
(2) If $n_{4^+}(v_2) = 1$, say $d(y_1) \geqslant 4$, then $y_1 \in F_3$.

Proof. By the minimality of G, $G - v$ has an (F_3, F_5)-partition. By Lemma 6.3, v_1 is F_3-saturated and v_2 is F_5-saturated in $G - v$.

(1) Suppose otherwise that $x_1 \in F_3$. Then exactly one vertex of $x_2, \ldots, x_4$ belongs to F_5. Since x_2, x_3, x_4 have degree at most 3, we can change v_1 to F_5 and then add v to F_3 to obtain an (F_3, F_5)-partition of G, a contradiction.

(2) Suppose otherwise that $y_1 \in F_5$. Similarly, exactly one vertex of $y_2, \ldots, y_6$ belongs to F_3. Again since $y_2, \ldots, y_6$ have degree at most 3, we can change v_2 to F_3 and then add v to F_5 to reach an (F_3, F_5)-partition of G, a contradiction. □

Lemma 6.5. *Every* 3*-vertex v has at least one* 5^+*-neighbour.*

Proof. Suppose to the contrary that all v's neighbours, denoted by v_1, v_2 and v_3, are of degree at most 4. Then, by the minimality of G, $G - v$ admits an (F_3, F_5)-partition. If $v_1, v_2, v_3 \in F_i$ for some fixed $i \in \{3, 5\}$, then we could add v to F_j, where $j \in \{3, 5\} \backslash \{i\}$. Otherwise, we assume that there exists some $i \in \{3, 5\}$, so that exactly one vertex of the set $\{v_1, v_2, v_3\}$ belongs to F_i. W.l.o.g., assume that $v_1 \in F_i$. Then, we put v to F_i. Since $d(v_1) \leqslant 4$, if the resultant partition of G is not an (F_3, F_5)-partition, then $i = 3$ and v_1 is F_3-saturated in $G - v$. It suffices to further change v_1 to F_5, and thus obtain an (F_3, F_5)-partition of G, a contradiction. □

Lemma 6.6. *Let v be a* 5*-vertex with $n_2(v) + n_{3^l}(v) \geqslant 1$. Then $n_{7^+}(v) \geqslant 1$.*

Proof. Let v be adjacent to $v_1, v_2, \ldots, v_5$ such that v_1 is either a 2-vertex or a light 3-vertex. Suppose to the contrary that $d(v_i) \leqslant 6$ for each $i \in \{2, 3, 4, 5\}$. By the minimality of G, $G - v_1$ admits an (F_3, F_5)-partition.

Case 1: Assume that v_1 is a 2-vertex.

Denote by v_1' the other neighbour of v_1. By Lemma 6.3, v is F_3-saturated in $G - v_1$ and $v_1' \in F_5$. W.l.o.g., assume that $v_2, v_3, v_4 \in F_3$ and $v_5 \in F_5$. Then, we change v to F_5, put v_1 to F_3, and further change v_5 to F_3 if it is F_5-saturated in $G - v$. One may check that the obtained partition is our desired partition, a contradiction.

Case 2: Assume that v_1 is a light 3-vertex.

Let v_1' and v_1'' denote the other two neighbours of v_1 different from v. By definition, $d(v_1') \leqslant 4$ and $d(v_1'') \leqslant 4$. In what follows, let $S = \{v, v_1', v_1''\}$ and $\{i, j\} = \{3, 5\}$. If all vertices of S belong to the same forest partition, say F_i, then we may put v_1 to F_j. Now assume that exactly one vertex of S is in F_i and the remaining two vertices of S are in F_j.

- If $v \in F_i$ and $v_1', v_1'' \in F_j$, then we add v_1 to F_i. If the obtained partition is not an (F_3, F_5)-partition, then we assert that $i = 3$, $j = 5$, and v is F_3-saturated in $G - v_1$. W.l.o.g., suppose that $v_k \in F_3$ for all $k = 2, 3, 4$ and $v_5 \in F_5$. At this moment, we can change v to F_5, and then change v_5 to F_3 if it is F_5-saturated in $G - v_1$, ensuring that the obtained partition of G is an (F_3, F_5)-partition, a contradiction.
- Now, by symmetry, assume that $v_1' \in F_i$ and $v, v_1'' \in F_j$. Firstly, add v_1 to F_i. If the resultant partition is not our wanted, then $i = 3$ and v_1' is F_3-saturated in $G - v_1$. It follows that $d(v_1') = 4$. Then we may further change v_1' to F_5 to get an (F_3, F_5)-partition of G, a contradiction. □

In the following, we say a vertex v is an i^j*-vertex* if $i \leqslant d(v) \leqslant j$.

Lemma 6.7. *Let v be a 6^9-vertex. If $n_{4^+}(v) = 0$, then $n_{3^h}(v) \geqslant 2$.*

Proof. Let $N_G(v) = \{v_1, v_2, \ldots, v_k\}$ with $6 \leqslant k \leqslant 9$. By assumption, $d(v_i) \leqslant 3$ for all $i \in \{1, 2, \ldots, k\}$. If $d(v_i) = 2$, then we use v_i' to denote the other neighbour of v_i. If $d(v_i) = 3$, then let v_i', v_i'' denote the other two neighbours (distinct to v) of v_i. Suppose to the contrary that $n_{3^h}(v) \leqslant 1$. W.l.o.g., assume that v_i is either a 2-vertex or a light 3-vertex for each $i \in \{2, 3, \ldots, k\}$.

Clearly, $G - \{v, v_2, \ldots, v_k\}$ has an (F_3, F_5)-partition due to the minimality of G. Firstly, let $2 \leqslant i \leqslant k$, $d, d' \in \{3, 5\}$ and $d \neq d'$. If $d(v_i) = 2$, then we add v_i to F_d when $v_i' \in F_{d'}$. If $d(v_i) = 3$ and $v_i', v_i'' \in F_d$, then add v_i to $F_{d'}$. Next, we have to count the number of vertices among $v_1, \ldots, v_k$ being in F_5.

If there are at most five F_5-vertices in $N_G(v)$, then we add v to F_5, and then add all remaining 3-vertices to F_3. Notice that it may occur some conflicts if $v_j \in F_3$ for $2 \leqslant j \leqslant k$ and v_j' or v_j'' is F_3-saturated. In this case, we may further change v_j' or v_j'' to F_5 and thus obtain an (F_3, F_5)-partition of G, a contradiction. Otherwise, we deduce that at least six F_5-vertices are in $N_G(v)$. This implies that there are at most three F_3-vertices in $N_G(v)$ due to $k \leqslant 9$. Hence, it suffices to add v to F_3 and add all remaining 3-vertices to F_5, leading to a contradiction. □

6.2.2 Structure of 5-faces

For convenience, we use $S_{F_i}(v)$ to denote the set of F_i-neighbours of v, where $i \in \{3, 5\}$.

Lemma 6.8. *Suppose that $f = [v_1 \dots v_5]$ is a weak 5-face such that $d(v_1) = 2$, $d(v_2) = 5$, $d(v_3) = d(v_4) = 3$ and v_3 is light. Then v_4 is a heavy 3-vertex.*

Proof. Suppose to the contrary that v_4 is light and let v_3' and v_4' be the third neighbour of v_3 and v_4, respectively. By Lemma 6.1, $d(v_5) \geqslant 7$, and thus we have that $d(v_3') \leqslant 4$ and $d(v_4') \leqslant 4$. Let $G' = G - \{v_3\}$. By the minimality of G, G' admits an (F_3, F_5)-partition. If $|S_{F_5}(v_3)| = 0$ or $|S_{F_3}(v_3)| = 0$, then we add v_3 to F_5 and F_3, respectively. If $|S_{F_5}(v_3)| = 1$, then add v_3 to F_5 easily since all its neighbours are of degree at most 5. Next, consider the remaining case that $|S_{F_5}(v_3)| = 2$. There are three possibilities below:

- If $v_2, v_3' \in F_5$ and $v_4 \in F_3$, then add v_3 to F_3 easily without causing any conflicts.
- If $v_2, v_4 \in F_5$ and $v_3' \in F_3$, then add v_3 to F_3 and further change v_3' to F_5 if it is F_3-saturated in G'.
- Now assume that $v_3', v_4 \in F_5$ and $v_2 \in F_3$. Then one of v_4' and v_5 belongs to F_5 since otherwise add v_3 to F_5 easily. Similarly, one of v_4' and v_5 belongs to F_3 since otherwise first change v_4 to F_3 and then go back to previous case. If $v_4' \in F_3$ and $v_5 \in F_5$, then change v_4 to F_3 and further change v_4' to F_5 if it is F_3-saturated in G', and then go back to the former case. Or else, assume $v_4' \in F_5$ and $v_5 \in F_3$. In this case, it remains us to change v_1 to F_5, add v_3 to F_3, and finally change v_2 to F_5 if it is F_3-saturated in G'. One can easily check that the obtained partition of G is an (F_3, F_5)-partition, a contradiction. □

Lemma 6.9. *Suppose that $f = [v_1 \dots v_5]$ is a bad 5-face such that $d(v_1) = 2$, $d(v_2) = 7$, $d(v_5) = 5$ and $n_{5^+}(v_5) = 1$. Then $n_{4^+}(v_2) \geqslant 2$.*

Proof. Let $N_G(v_2) = \{v_1, v_3, u_1, \dots, u_5\}$ and $N_G(v_5) = \{v_1, v_4, w_1, w_2, w_3\}$. Suppose otherwise that v_2 has at most one 4^+-neighbour. Let $G' = G - \{v_1\}$. Then G' admits an (F_3, F_5)-partition due to the minimality of G. By Lemma 6.3, $v_2 \in F_5$ and v_2 has exactly five F_5-neighbours, and $v_5 \in F_3$ and v_5 has exactly three F_3-neighbours. Since f is bad, exactly one of v_3 and v_4 is of degree 2.

First suppose that $d(v_4) = 2$. By Lemma 6.1, $d(v_3) \geqslant 7$. It follows that all vertices of $u_1, \dots, u_5$ are 3^--vertices by assumption. W.l.o.g., let w_1 be the 5^+-neighbour of v_5. By Lemma 6.4 (2), we know that $v_3 \in F_3$. If $w_1 \in F_3$, then we could change v_5 to F_5 and add v_1 to F_3 since w_2, w_3, v_4 have degree at most 4. So $w_1 \in F_5$, implying that w_2, w_3, v_4 are in F_3 and all $u_1, \dots, u_5$ are in F_5. We only need to change v_4 to F_5 and then add v_1 to F_3.

Now suppose that $d(v_3) = 2$. Then $d(v_4) \geqslant 5$ by Lemma 6.1, which implies that all w_1, w_2, w_3 are 4^--vertices. By assumption, we may let u_1 be the 4^+-neighbour of v_2 if it exists. If $v_4 \in F_3$, then exactly one of w_1, w_2, w_3 is in F_5, say w_1. One may first add v_1 to F_3, and then change v_5 to F_5. Next, assume that $v_4 \in F_5$. Recall that v_2 has exactly five F_5-neighbours. If $v_3 \in F_3$, then all $u_1, \ldots, u_5$ are in F_5, and so we only need to change v_2 to F_3 and then add v_1 to F_5. If $v_3 \in F_5$, then it suffices to change v_3 to F_3 and then go back to the previous case. □

Lemma 6.10. *Suppose that $f = [v_1 \ldots v_5]$ is a bad 5-face such that $d(v_1) = 2$, $d(v_2) = 8$ and $d(v_5) = 5$. If $d(v_4) = 2$ and $n_{4^+}(v_2) = 1$, then $n_3(v_2) \geqslant 1$.*

Proof. Let $N_G(v_2) = \{v_1, v_3, u_1, \ldots, u_6\}$. Let $G' = G - \{v_1\}$. By the minimality of G, G' admits an (F_3, F_5)-partition. Again, by Lemma 6.3, we know that v_2 is F_5-saturated and v_5 is F_3-saturated.

Suppose to the contrary that $d(u_i) = 2$ for all $i \in \{1, \ldots, 6\}$. By Lemma 6.1, $d(v_3) \geqslant 7$. If $v_4 \in F_5$, then $w_i \in F_3$ for each $i \in \{1, 2, 3\}$. We can add v_1 to F_3 and then change v_5 to F_5, a contradiction. Next assume $v_4 \in F_3$. If $v_3 \in F_3$, then change v_4 to F_5 and then reduce to the above case. Otherwise, assume that $v_3 \in F_5$. Since v_2 has exactly five F_5-neighbours, we may let u_1 and u_2 be in F_3. Let $N_G(u_1) = \{v_2, u_1'\}$ and $N_G(u_2) = \{v_2, u_2'\}$. At this moment, one may first change v_2 to F_3, and then add v_1 to F_5. If the resultant partition is not our desired partition, we may deduce that both u_1' and u_2' are in F_3. In this case, it suffices to continue to change both u_1 and u_2 to F_5, a contradiction. □

6.2.3 Structure of adjacent 5-faces

Lemma 6.11. *Suppose $f = [v_1 \ldots v_5]$ and $g = [v_1 v_5 \ldots v_8]$ are adjacent weak 5-faces such that $d(v_2) = d(v_8) = 2$, $d(v_4) = d(v_5) = d(v_6) = 3$ and $d(v_1) = 6$. Then at least one of v_4 and v_6 is a heavy 3-vertex.*

Proof. Since $g(G) \geqslant 5$, we have that $|V(f) \cap V(g)| = 2$. That is, $\{v_2, v_3, v_4\} \cap \{v_6, v_7, v_8\} = \varnothing$, as shown in Figure 6.3. Let v_4' and v_6' denote the third neighbour of v_4 and v_6 not on the boundary of f and g, respectively. By Lemma 6.1, we know that both v_3 and v_7 are 7^+-vertices. Suppose to the contrary that neither v_4 nor v_6 is a heavy 3-vertex. It follows that v_4' and v_6' are both 4^--vertices. Due to $d(v_1) = 6$, let w_1, w_2, w_3 denote the other neighbours of v_1 different from v_2, v_5 and v_8. $G - \{v_5\}$ has an (F_3, F_5)-partition by the minimality of G.

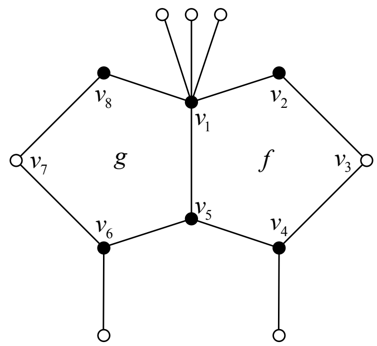

Figure 6.3: The configuration of Lemma 6.11.

If $|S_{F_5}(v_5)| = 0$ or $|S_{F_3}(v_5)| = 0$, then we add v_5 to F_5 and F_3 directly, respectively. If $|S_{F_5}(v_5)| = 1$, then we put v_5 to F_5. If the obtained partition of G is not our desired partition, then it should be the case that $v_1 \in F_5$, $v_4, v_6 \in F_3$, and v_1 is F_5-saturated in $G - \{v_5\}$. In this case, we only need to change v_1 to F_3 and thus get an (F_3, F_5)-partition of G. Now consider the last case that $|S_{F_5}(v_5)| = 2$. If exactly one of v_4 and v_6 belongs to F_5, say $v_4 \in F_5$, then it is easy to add v_5 to F_3. So next, assume that $v_4, v_6 \in F_5$ and $v_1 \in F_3$. There are four subcases below.

- $v_3, v_4' \in F_3$. We can add v_5 to F_5 directly.
- $v_3, v_4' \in F_5$. We first change v_4 to F_3 and then add v_5 to F_5.
- $v_3 \in F_5$ and $v_4' \in F_3$. We first change v_4 to F_3, add v_5 to F_5, and then change v_4' to F_5 if it is F_3-saturated in $G - \{v_5\}$.
- $v_3 \in F_3$ and $v_4' \in F_5$. By symmetry, suppose that $v_7 \in F_3$ and $v_6' \in F_5$. In this case, one may change v_2 and v_8 both to F_5. If at most two of w_1, w_2, w_3 are in F_3, then we further add v_5 to F_3. Or else, we can change v_1 to F_5 and then add v_5 to F_3. □

Lemma 6.12. *Suppose $f = [vv_1u_1u_2v_2]$ and $g = [v_1vv_2u_3u_4]$ are adjacent bad 5-faces such that $d(v) = 2$ and $d(v_1) = 5$. If one of the following conditions holds, then $d(v_2) \geqslant 8$.*
(1) $d(u_1) = d(u_4) = 2$ and $n_{4^+}(v_2) = 2$;
(2) $d(u_2) = d(u_3) = 2$ and $n_{4^+}(v_1) = 2$.

Proof. By Lemma 6.1, $d(v_2) \geqslant 7$. In each of following cases, suppose to the contrary that $d(v_2) = 7$. Let $N_G(v_1) = \{v, u_1, u_4, w_1, w_2\}$ and $N_G(v_2) = \{v, u_2, u_3, x_1, \ldots, x_4\}$. Let $G' = G - \{v\}$. Clearly, by the minimality of G, G' has an (F_3, F_5)-partition. By Lemma 6.3, we know that $v_1 \in F_3$ and $v_2 \in F_5$. Moreover, v_1 has exactly three F_3-neighbours and v_2 has exactly five F_5-neighbours.

(1) Suppose that $d(u_1) = d(u_4) = 2$ and $n_{4^+}(v_2) = 2$. Then $d(u_2) \geqslant 7$ and $d(u_3) \geqslant 7$ by Lemma 6.1. Since v_2 has exactly two 4^+-neighbours, we know that all $x_1, \ldots, x_4$ are 3^--vertices. Observe that at least one of u_2 and u_3 belongs to F_5 because v_2 is an F_5-saturated vertex. If both u_2 and u_3 are in F_5, then exactly one of $x_1, \ldots, x_4$ is in

F_3, say $x_1 \in F_3$. In this case, we can change v_2 to F_3 and then add v to F_5. Now by symmetry, suppose that $u_2 \in F_3$ and $u_3 \in F_5$. This guarantees that $x_1, \ldots, x_4$ are all in F_5. If $u_1 \in F_3$, then change u_1 to F_5 and then add v to F_3. Or else, assume that $u_1 \in F_5$. Then all w_1, w_2, u_4 belong to F_3, and therefore one can first change v_1 to F_5 and then add v to F_3. One may always obtain a contradiction.

(2) Suppose that $d(u_2) = d(u_3) = 2$ and $n_{4^+}(v_1) = 2$. By Lemma 6.1, both u_1 and u_4 are 5^+-vertices. Since v_1 has exactly two 4^+-neighbours, we know that w_1 and w_2 are both 3^--vertices. If some $w_i \in F_5$ with $i \in \{1, 2\}$, then change v_1 to F_5 and then add v to F_3. So next, by symmetry, assume that $u_1 \in F_5$. If $u_2 \in F_3$, then change v_2 to F_3 and add v to F_5. Otherwise, we change u_2 to F_3 and add v to F_5. In both cases, one can always obtain an (F_3, F_5)-partition of G, a contradiction. □

Lemma 6.13. *Suppose $f = [vv_1u_1u_2v_2]$ and $g = [v_1vv_2u_3u_4]$ are adjacent bad 5-faces such that $d(v) = d(u_1) = d(u_4) = 2$, $d(v_1) = 5$ and $n_{4^+}(v_2) = 2$.*
(1) If $d(v_2) = 8$, then $n_3(v_2) \geqslant 2$;
(2) If $d(v_2) = 9$, then $n_3(v_2) \geqslant 1$.

Proof. Let $N_G(v_2) = \{v, u_2, u_3, x_1, \ldots, x_k\}$ with $k \geqslant 5$. Since $d(u_1) = d(u_4) = 2$, both u_2 and u_3 have degree at least 7 by Lemma 6.1, and thus all vertices $x_1, \ldots, x_k$ are 3^--vertices. Let $G' = G - \{v\}$. Then G' admits an (F_3, F_5)-partition by the minimality of G. Again, by Lemma 6.3, v_1 is F_3-saturated and v_2 is F_5-saturated. If $u_i \in F_5$ for some $i \in \{1, 4\}$, then we can change v_1 to F_5 and then add v to F_3. Now suppose both u_1 and u_4 are in F_3. Similarly, if $u_i \in F_3$ for some $i \in \{2, 3\}$, say u_2, then we can change u_1 to F_5 and then add v to F_3. In what follows, suppose $u_2 \in F_5$ and $u_3 \in F_5$. It means that exactly three of $x_1, \ldots, x_k$ belong to F_5.

(1) Suppose otherwise that $n_3(v_2) \leqslant 1$. Namely at least four vertices among $x_1, \ldots, x_5$ are of degree exactly 2. If one cannot change v_2 to F_3 without any conflicts, then there exists some x_i with $i \in \{1, \ldots, 5\}$ such that $d(x_i) = 2$ and $x_i \in F_3$, say x_1. Let $N_G(x_1) = \{v_2, x_1'\}$. Notice that $x_1' \in F_3$. Then we continue to change x_1 to F_5, and then add v to F_5. It is easy to check that the obtained partition is an (F_3, F_5)-partition of G.

(2) Suppose otherwise that $n_3(v_2) = 0$. It follows that $x_1, \ldots, x_6$ are all 2-vertices. For each $i \in \{1, \ldots, 6\}$, let x_i' denote the other neighbour of x_i distinct to v_2. By a similar discussion as above, we see that exactly three vertices of $x_1, \ldots, x_6$ belong to F_3, say x_1, x_2 and x_3. If we are not able to change v_2 to F_3, then there exist at least two of x_1', x_2' and x_3' being in F_3, say x_1' and x_2'. Thus, it remains us to continue to change x_1 and x_2 to F_3, and finally add v to F_5, a contradiction. □

Lemma 6.14. *Suppose $f = [vv_1u_1u_2v_2]$ and $g = [v_1vv_2u_3u_4]$ are adjacent bad 5-faces such that $d(v) = d(u_2) = d(u_3) = 2$, $d(v_1) = 5$ and $n_{4^+}(v_1) = 2$. Then the following holds.*
(1) $d(u_i) \geqslant 8$ for some $i \in \{1, 4\}$;
(2) If $d(u_1) = 8$ and $d(u_4) \leqslant 7$, then $n_{4^+}(u_1) \geqslant 2$.

Proof. Let w_1 and w_2 denote the other two neighbours of v_1 different from u_1, u_4 and v. By Lemma 6.1, we have that $d(u_1) \geqslant 5$, $d(u_4) \geqslant 5$ and $d(v_2) \geqslant 7$. Since $n_{4^+}(v_1) = 2$, both w_1 and w_2 are 3^--vertices. In the following, let $G' = G - \{v\}$. Then by the minimality of G, G' admits an (F_3, F_5)-partition. By Lemma 6.3, v_1 is F_3-saturated and v_2 is F_5-saturated. If either w_1 or w_2 belongs to F_5, then we change v_1 to F_5 directly, and further add v to F_3. Next, assume that $w_1, w_2 \in F_3$, implying that exactly one of u_1 and u_4 is in F_5.

(1) Suppose to the contrary that both u_1 and u_4 are 7^--vertices. By symmetry, let $u_1 \in F_5$. Then $u_4 \in F_3$. If $u_2 \in F_5$, then change u_2 to F_3, and add v to F_5. Or else, assume that $u_2 \in F_3$. If there are still some conflicts to change v_1 to F_5, then u_1 has already five F_5-neighbours distinct to v_1. It follows that $d(u_1) = 7$. In this case, one can first change u_1 to F_3, v_1 to F_5, and finally add v to F_3, a contradiction.

(2) Since $d(v_1) = 5$, we may suppose to the contrary that all neighbours of u_1 not on f, say $x_1, \ldots, x_6$, are of degree at most 3.

Consider the first case that $u_1 \in F_5$. Then $u_4 \in F_3$. If $u_2 \in F_5$, then we change u_2 to F_3, and add v to F_5. Otherwise, $u_2 \in F_3$. If we failed to change v_1 to F_5, then u_1 must be an F_5-saturated vertex in G'. Namely, exactly five vertices among $x_1, \ldots, x_6$ are in F_5 and the remaining one is in F_3 since $d(u_1) = 8$. At this moment, we can change u_1 to F_3, v_1 to F_5 and finally add v to F_3.

Now consider the case that $u_4 \in F_5$. Then $u_1 \in F_3$. If $u_3 \in F_5$, then we change u_3 to F_3 and add v to F_5 immediately. Otherwise, assume $u_3 \in F_3$. Since $d(u_4) \leqslant 7$, it is easy to establish an (F_3, F_5)-partition of G by changing v_1 to F_5, adding v to F_3, and changing u_4 to F_3 if it is F_5-saturated in G'.

In each case, one may verify that the obtained partition is an (F_3, F_5)-partition of G, a contradiction. □

Lemma 6.15. *Suppose $f = [vv_1u_1u_2v_2]$ and $g = [v_1vv_2u_3u_4]$ are adjacent bad 5-faces such that $d(v) = d(u_2) = d(u_4) = 2$, $d(v_1) = 5$ and $d(v_2) = 7$. Then for each $i \in \{1, 2\}$, we have that $n_{4^+}(v_i) \geqslant 2$.*

Proof. By Lemma 6.1, $d(u_1) \geqslant 5$ and $d(u_3) \geqslant 7$. Let $G' = G - \{v\}$. Then by the minimality of G, G' admits an (F_3, F_5)-partition. Let w_1, w_2 and $x_1, \ldots, x_4$ denote the other neighbours of v_1 and v_2, respectively. By Lemma 6.3, v_1 is F_3-saturated and v_2 is F_5-saturated in G'.

First suppose to the contrary that v_1 has exactly one 4^+-neighbour. That is, u_1 is such a vertex. It follows that $d(w_i) \leqslant 3$ for $i \in \{1, 2\}$. By Lemma 6.4 (1), we see that $u_1 \in F_5$. If $u_2 \in F_5$, then we can change u_2 to F_3 and then add v to F_5 successfully. Otherwise, assume that $u_2 \in F_3$. It means that all remaining neighbours of v_2 that are $x_1, \ldots, x_4$ and u_3, belong to F_5. We only need to further change v_2 to F_3 and then add v to F_5, a contradiction.

Now suppose to the contrary that v_2 has exactly one 4^+-neighbour. Namely, u_3 is such a vertex. Then $d(x_i) \leqslant 3$ for all $i \in \{1, \ldots, 4\}$. By Lemma 6.4 (2), $u_3 \in F_3$. If $u_4 \in F_3$, then we can change u_4 to F_5 and further add v to F_3. Otherwise, assume that $u_4 \in F_5$. It suffices to change v_1 to F_5 and then add v to F_3 successfully. □

Next, for simplicity, in Lemmas 6.16-6.19, we will use f_e to denote the face that is adjacent to f by the common edge e.

Lemma 6.16. *Suppose $f = [vv_1u_1u_2v_2]$ and $g = [v_1vv_2u_3u_4]$ are adjacent bad 5-faces such that $d(v) = d(u_2) = d(u_4) = 2$, $d(v_1) = 5$, $d(v_2) = 7$, $n_{4^+}(v_1) = 2$ and $n_{4^+}(v_2) = 2$. Let $f_{u_2v_2} = [u_1u_2v_2x_1y]$ be a bad 5-face. Then we have:*
(1) If $d(y) = 2$, then for each $t \in \{u_1, x_1\}$, we have $d(t) \geqslant 6$; moreover, $n_{4^+}(t) \geqslant 2$ if $d(t) = 6$.
(2) Suppose $d(x_1) = 2$.
(2.1) If $d(u_1) = 5$, then $n_{4^+}(u_1) \geqslant 3$;
(2.2) If $d(y) = 5$ and the face adjacent to $f_{u_2v_2}$ by common edges yx_1 and x_1v_2 is bad, then $n_{4^+}(y) \geqslant 3$.

Proof. Let $N_G(v_1) = \{v, u_1, u_4, w_1, w_2\}$ and $N_G(v_2) = \{v, u_2, u_3, x_1, \ldots, x_4\}$. By Lemma 6.1, $d(u_1) \geqslant 5$ and $d(u_3) \geqslant 7$. Since $n_{4^+}(v_1) = 2$, w.l.o.g., assume that $d(w_1) \geqslant 4$. Let $G' = G - \{v\}$. By the minimality of G, G' admits an (F_3, F_5)-partition. By Lemma 6.3, v_1 is F_3-saturated and v_2 is F_5-saturated. If $u_4 \in F_5$, then change v_1 to F_5 and add v to F_3. So $u_4 \in F_3$. If $u_3 \in F_3$, then change u_4 to F_5 and go back to the previous case. It follows that $u_3 \in F_5$. If $u_2 \in F_3$, then it is easy to change v_2 to F_3 and then add v to F_5. Therefore, $u_2 \in F_5$. Here, one may further deduce that $u_1 \in F_3$ since otherwise we can change u_2 to F_3 and reduce to the former case.

(1) Suppose $d(y) = 2$. By Lemma 6.1, $d(x_1) \geqslant 5$. This means that all remaining vertices x_2, x_3, x_4 are 3^--vertices due to $n_{4^+}(v_2) = 2$. If $x_1 \in F_5$, then we can easily change v_2 to F_3 and then add v to F_5 successfully. So next assume $x_1 \in F_3$. This implies that u_2, x_2, x_3, x_4 all belong to F_5. If $y \in F_3$, then change y to F_5, u_2 to F_3, and finally add v to F_5. In what follows, assume that $y \in F_5$.

We first shall prove that $d(t) \geqslant 6$ for each $t \in \{u_1, x_1\}$. If $d(u_1) = 5$, let y_1, y_2 denote its other two neighbours. Then both y_1, y_2 are in F_3 since otherwise we can change u_2 to F_3 and then add v to F_5 successfully. This fact enables us to change u_1 to F_5 and then add v to F_3, a contradiction. So $d(u_1) \geqslant 6$. Similarly, if $d(x_1) = 5$, let z_1, z_2, z_3 denote its other three neighbours, then we can change v_2 to F_3 and add v to F_5. If the resultant partition is not our desired partition, it should be the case that x_1 is F_3-saturated in G'. That is, all z_1, z_2, z_3 belong to F_3. Thus, we have to further change x_1 to F_5 to reach an (F_3, F_5)-partition of G, a contradiction. Hence, $d(x_1) \geqslant 6$.

Next we shall prove that for each $t \in \{u_1, x_1\}$, $n_{4^+}(t) \geqslant 2$ if $d(t) = 6$. First suppose to the contrary that $d(u_1) = 6$ and $n_{4^+}(u_1) = 1$. It means that each vertex in $N_G(u_1)\backslash\{v_1, u_2, y\}$ is a 3^--vertex. Since $|N_G(u_1)\backslash\{v_1, u_2, y\}| = 3$, we first observe that u_1 must be F_3-saturated, since otherwise we can change u_2 to F_3 and then add v to F_5, a contradiction. This implies that exactly one vertex in $N_G(u_1)\backslash\{v_1, u_2, y\}$ belongs to F_5. Therefore, we can change u_1 to F_5, u_2 to F_3, and then add v to F_3, a contradiction. Now suppose to the contrary that $d(x_1) = 6$ and $n_{4^+}(x_1) = 1$. Then $|N_G(x_1)\backslash\{v_2, y\}| = 4$ and all vertices in $N_G(x_1)\backslash\{v_2, y\}$ have degree at most 3. We deduce that x_1 is F_3-saturated, since otherwise we can change v_2 to F_3 and then add v to F_5, a contradiction. This also implies that exactly one vertex of $N_G(x_1)\backslash\{v_2, y\}$ belongs to F_5. Hence, it is easy to change x_1 to F_5, v_2 to F_3 and then add v to F_5. One can verify that the obtained partition is our desired partition, a contradiction.

(2) Suppose that $d(x_1) = 2$. By Lemma 6.1, $d(y) \geqslant 5$. It is obvious that $x_1 \in F_5$, since otherwise we can change v_2 to F_3 and then add v to F_5, a contradiction. If $y \in F_5$, then we can change x_1 to F_3 and then add v to F_5 easily. Next, we can assume that $y \in F_3$.

- Assume $d(u_1) = 5$. Suppose otherwise that u_1 has at most two 4^+-neighbours. That is, $d(y_1) \leqslant 3$ and $d(y_2) \leqslant 3$, where $y_1, y_2 \in N_G(u_1)\backslash\{v_1, u_2, y\}$. If we can change u_2 to F_3, then it is easy to obtain an (F_3, F_5)-partition by further adding v to F_5. Otherwise,it must be the case that u_1 is F_3-saturated. Namely, exactly one of y_1, y_2 belongs to F_3, and thus exactly one of y_1, y_2 belongs to F_5. Therefore, we can change u_1 to F_5, and then add v to F_3, a contradiction. Hence, $n_{4^+}(u_1) \geqslant 3$.

- Assume $d(y) = 5$. Suppose to the contrary that $n_{4^+}(y) \leqslant 2$. Let y_1, y_2, y_3 denote the other three neighbours of y distinct to u_1 and x_1. Let $f' = [yx_1v_2x_2y_1]$ be a bad 5-face. Observe that y is F_3-saturated since otherwise we can change x_1 to F_3 and then add v to F_5 successfully. It guarantees us that exactly two of y_1, y_2, y_3 belong to F_3 and the remaining one belongs to F_5. Since f' is bad, we see that either x_2 or y_1 is a 2-vertex.
  - If $d(x_2) = 2$, then $d(y_1) \geqslant 5$ by Lemma 6.1. Then $d(y_i) \leqslant 3$ for both $i = 2, 3$. One may deduce that $x_2 \in F_5$; if not, we can change v_2 to F_3 and then add v to F_5, a contradiction. This would imply that $y_1 \in F_3$, since otherwise we can change x_2 to F_3 and then add v to F_5, a contradiction. Therefore, we can change y to F_5, x_1 to F_3 and then add v to F_5.
  - If $d(y_1) = 2$, then $d(x_2) \geqslant 7$ by Lemma 6.1. We deduce that $x_2 \in F_3$, since otherwise we can change v_2 to F_3 and then add v to F_5, a contradiction. If $y_1 \in F_5$, then change y to F_5, x_1 to F_3, and finally add v to F_5. Otherwise, we may first change y_1 to F_5 and then go back to the former case. □

Lemma 6.17. *Suppose $f = [vv_1u_1u_2v_2]$ and $g = [v_1vv_2u_3u_4]$ are adjacent bad 5-faces such that $d(v) = d(u_2) = d(u_4) = 2$, $d(v_1) = 5$ and $d(v_2) = 8$. Then we have:*
(1) If $n_{4^+}(v_1) = 1$, then $n_{4^+}(v_2) \geqslant 2$.
(2) Suppose $n_{4^+}(v_1) = 2$ and $n_{4^+}(v_2) = 1$. Then
(2.1) $n_3(v_2) \geqslant 2$;
(2.2) Let $f_{u_2v_2} = [u_1u_2v_2x_1y]$ be a bad 5-face.
(2.2.1) If $d(u_1) = 5$, then $n_{4^+}(u_1) \geqslant 3$;
(2.2.2) If $d(y) = 5$ and the face adjacent to $f_{u_2v_2}$ by common edges yx_1 and x_1v_2 is bad, then $n_{4^+}(y) \geqslant 3$.

Proof. Let $N_G(v_1) = \{v, u_1, u_4, w_1, w_2\}$ and $N_G(v_2) = \{v, u_2, u_3, x_1, \ldots, x_5\}$. Then $d(u_1) \geqslant 5$ and $d(u_3) \geqslant 7$ by Lemma 6.1. By the minimality of G, $G - \{v\}$ admits an (F_3, F_5)-partition. By Lemma 6.3, v_1 is F_3-saturated and v_2 is F_5-saturated. If $u_4 \in F_5$, then change v_1 to F_5 and add v to F_3. So $u_4 \in F_3$. If $u_3 \in F_3$, then change u_4 to F_5 and then add v to F_3. Next, assume that $u_3 \in F_5$. If $u_2 \in F_3$, then we change v_2

to F_3, change u_2 to F_5 if $u_1 \in F_3$, and then add v to F_5. Thus, $u_2 \in F_5$. Moreover, one may further deduce that $u_1 \in F_3$.

(1) Suppose to the contrary that $n_{4^+}(v_2) \leqslant 1$. That is, all $x_1, \dots, x_5$ are 3^--vertices. Since $n_{4^+}(v_1) = 1$, we know that $d(w_i) \leqslant 3$ for both $i \in \{1, 2\}$. Thus, we can first change v_1 to F_5, and then add v to F_3 successfully, a contradiction.

(2) Since $n_{4^+}(v_1) = 2$ and $n_{4^+}(v_2) = 1$, let $d(w_1) \geqslant 4$ and $d(x_i) \leqslant 3$ for all $i \in \{1, \dots, 5\}$. By a similar way, it is not difficult to obtain that $u_2 \in F_5$ and $u_1 \in F_3$.

(2.1) Suppose otherwise that $n_3(v_2) \leqslant 1$. W.l.o.g., assume that $d(x_i) = 2$ for all $i \in \{1, 2, 3, 4\}$. If v_2 cannot be successfully changed to F_3, it must be the case that two vertices among $x_1, \dots, x_5$ are in F_3, let $x_1, x_i \in F_3$, where $i \in \{2, 3, 4, 5\}$, so that the unique neighbour of x_1, denoted by x_1', belongs to F_3. In this case, we only need to further change x_1 to F_5, v_2 to F_3, and finally add v to F_5, a contradiction.

(2.2) Since $d(x_1) \leqslant 3$, by Lemma 6.1, we affirm that $d(x_1) = 2$ and $d(y) \geqslant 5$. Similarly, we deduce that $x_1 \in F_5$ and $y \in F_3$.

(2.2.1) Suppose otherwise that $n_{4^+}(u_1) = 2$. Then the remaining two neighbours of u_1 not on f, say y_1, y_2, are of degree at most 3. Notice that exactly one of y_1, y_2 belongs to F_5 by Lemma 6.3. Thus, we can change u_1 to F_5 and then add v to F_3.

(2.2.2) Suppose to the contrary that $n_{4^+}(y) \leqslant 2$. Let $f' = [yx_1v_2x_2z]$ be the bad 5-face adjacent to $f_{u_2v_2}$. By Lemma 6.1, $d(x_2) = 2$ and $d(z) \geqslant 5$. So the other two neighbours of y different from x_1, u_1, z, denoted by t_1, t_2, are 3^--vertices. Similarly, $x_2 \in F_5$ and $z \in F_3$. Moreover, y is F_3-saturated by Lemma 6.3. This ensures us that one of t_1, t_2 belongs to F_5. So we can obtain an (F_3, F_5)-partition of G by changing y to F_5, x_1 to F_3 and finally adding v to F_5, a contradiction. □

Lemma 6.18. *Suppose $f = [vv_1u_1u_2v_2]$ and $g = [v_1vv_2u_3u_4]$ are adjacent bad 5-faces such that $d(v) = d(u_2) = d(u_4) = 2$, $d(v_1) = 5$ and $d(v_2) = 9$. If $n_{4^+}(v_i) = 1$ for both $i \in \{1, 2\}$, then $n_3(v_2) \geqslant 2$.*

Proof. Let $N_G(v_1) = \{v, u_1, u_4, w_1, w_2\}$ and $N_G(v_2) = \{v, u_2, u_3, x_1, \dots, x_6\}$. By Lemma 6.1, $d(u_1) \geqslant 5$ and $d(u_3) \geqslant 7$. Since $n_{4^+}(v_1) = n_{4^+}(v_2) = 1$, we see that $d(w_i) \leqslant 3$ for $i \in \{1, 2\}$ and $d(x_j) \leqslant 3$ for $j \in \{1, \dots, 6\}$. Let $G' = G - \{v\}$. Then G' admits an (F_3, F_5)-partition. Again, by Lemma 6.3, we assert that v_1 is F_3-saturated and v_2 is F_5-saturated. Similarly, by an analogous discussion as above lemma, we derive that $u_4 \in F_3$ and $u_3 \in F_5$.

Suppose to the contrary that $n_3(v_2) \leqslant 1$. W.l.o.g., assume that $x_1, \dots, x_5$ are 2-vertices and x_6 is a 3^--vertex. Then $u_1 \in F_5$ since otherwise we can change v_1 to F_5 and then add v to F_3. If $u_2 \in F_5$, then we can change it to F_3 and then add v to F_5. So $u_2 \in F_3$. Because v_2 is an F_5-saturated vertex, there exist exactly four vertices among $x_1, \dots, x_6$ belonging to F_5, and therefore the remaining two vertices of $x_1, \dots, x_6$ are in F_3. W.l.o.g., assume that $x_1, x_i \in F_3$ such that $i \in \{2, \dots, 6\}$. One can change v_2 to F_3. If the resultant partition is not our wanted, then the unique neighbour of x_1 different from v_2 must be in F_3, and hence we only need to change x_1 to F_5 and then add v to F_5 successfully . □

Lemma 6.19. *Suppose $f = [vv_1u_1u_2v_2]$ and $g = [v_1vv_2u_3u_4]$ are adjacent bad 5-faces such that $d(v) = d(u_2) = d(u_4) = 2$, $d(v_1) = 6$ and $n_{4^+}(v_i) = 1$ for each $i \in \{1, 2\}$. Then*
(1) $d(v_2) \geqslant 8$;
(2) If $d(v_2) = 8$, $d(u_1) = 5$ and $n_{4^+}(u_1) = 2$, then $f_{u_2v_2}$ cannot be a bad 5-face.

Proof. Let $N_G(v_1) = \{v, u_1, u_4, w_1, w_2, w_3\}$ and $N_G(v_2) = \{v, u_2, u_3, x_1, \ldots, x_k\}$. Let $G' = G - \{v\}$. Then G' admits an (F_3, F_5)-partition due to the minimality of G. By Lemma 6.1, $d(u_1) \geqslant 5$ and $d(u_3) \geqslant 7$. Since $n_{4^+}(v_1) = n_{4^+}(v_2) = 1$, we see that $d(w_i) \leqslant 3$ for all $i \in \{1, 2, 3\}$ and $d(x_j) \leqslant 3$ for all $j \in \{1, \ldots, k\}$. Moreover, by Lemma 6.3, v_1 is F_3-saturated and v_2 is F_5-saturated.

(1) By Lemma 6.1, suppose to the contrary that $d(v_2) = 7$. That is, $k = 4$. If $u_3 \in F_5$, then exactly one of $u_2, x_1, \ldots, x_4$ belongs to F_3, which enables us to change v_2 to F_3 and then add v to F_5 successfully. Now assume that $u_3 \in F_3$. This means that $u_2, x_1, \ldots, x_4$ are all in F_5. If $u_1 \in F_5$, then we may change u_2 to F_3 and then add v to F_5. Otherwise, suppose $u_1 \in F_3$. If $u_4 \in F_3$, then change u_4 to F_5 and then add v to F_3. Or else, assume $u_4 \in F_5$. It is easy to obtain an (F_3, F_5)-partition by changing v_1 to F_5 and adding v to F_3, a contradiction.

(2) Suppose to the contrary that $f_{u_2v_2} = [u_1u_2v_2x_1y]$ is a bad 5-face. By Lemma 6.1, $d(x_1) = 2$ and $d(y) \geqslant 5$. Let $N_G(u_1) = \{v_1, u_2, y, y_1, y_2\}$. Then $d(y_1) \leqslant 3$ and $d(y_2) \leqslant 3$ due to the assumption that $n_{4^+}(u_1) = 2$. If $u_2 \in F_3$, then at most one of $x_1, \ldots, x_5$ belongs to F_3, and thus we can change v_2 to F_3, u_2 to F_5 if $u_1 \in F_3$, and finally add v to F_5. So $u_2 \in F_5$. This ensures us that $u_1 \in F_3$ since otherwise one may change u_2 to F_3 and then go back to the former case. Moreover, u_1 is F_3-saturated. By a similar way, we deduce that $x_1 \in F_5$ and $y \in F_3$. This implies that exactly one of y_1, y_2 is in F_5, and therefore one can firstly change u_1 to F_5, and then add v to F_3. □

6.3 Discharging procedure

In what follows, we will apply a discharging procedure to derive a contradiction. An *initial charge* ω on $V(G) \cup F(G)$ are defined as: $\omega(x) = d(x) - 4$ for every $x \in V(G) \cup F(G)$. By the relation $\sum_{v \in V(G)} d(v) = 2|E(G)|$ and Euler's formula, we see that the total sum of charge of the vertices and faces satisfies the following

$$\sum_{x \in V(G) \cup F(G)} \omega(x) = \sum_{x \in V(G) \cup F(G)} (d(x) - 4) = -8.$$

Note that any discharging procedure preserves the total sum of charges on G. So if we can define appropriate discharging rules to change the initial charge ω to the final charge ω^* on $V(G) \cup F(G)$ such that $\sum\limits_{x \in V(G) \cup F(G)} \omega^*(x) \geqslant 0$, then we have obtained a contradiction.

For $x \in V(G) \cup F(G)$, we use $\tau(x \to y)$ to denote the amount of charges transferring from x to y. Suppose that v is a heavy 3-vertex adjacent to v_1, v_2, and v_3 such that $d(v_1), d(v_2) \geqslant 5$ and v_3 is a 4^--vertex. Let f_i be the face incident to v by vv_i, vv_{i+1} as boundary edges, where indices are taken modulo 3. If $d(f_2) = 5$, then we call v a *special 3-vertex* of f_2. We usually use $n_{3^s}(f_2)$ to denote the number of special 3-vertices of f_2.

Below are the needed discharging rules:

(R0) Every heavy 3-vertex v sends $\frac{1}{6}$ to each light 3-neighbour u if one of the faces incident to uv is a 6^+-face.

(R1) Let v be a 5-vertex in G.

(R1.1) If $n_{4^+}(v) \geqslant 1$, then v sends $\frac{1}{n_{3^-}(v)}$ to each of 2-neighbour and light 3-neighbour;

(R1.2) Otherwise, v sends $\frac{1}{5}$ to each 3^--neighbour.

(R2) Let v be a 6^9-vertex.

(R2.1) If $n_{4^+}(v) \geqslant 1$, then

(R2.1a) when v is a 6^7-vertex, v sends $\frac{d(v)-4}{n_{3^-}(v)}$ to each 3^--neighbour;

(R2.1b) when v is an 8^9-vertex, v sends $\frac{1}{2}$ to each 3-neighbour and then distributes its excess charge of $d(v) - 4 - \frac{1}{2}n_3(v)$ uniformly among its 2-neighbours.

(R2.2) If $n_{4^+}(v) = 0$, then v sends $\frac{1}{6}$ to each heavy 3-neighbour and then distributes its excess charges $d(v) - 4 - \frac{1}{6}n_{3^h}(v)$ uniformly among its 2-neighbours and light 3-neighbours.

(R3) Every 10^+-vertex sends $\frac{d(v)-4}{n_{3^-}(v)}$ to each 3^--neighbour.

(R4) Let f be a 5-face.

(R4.1) If f is bad, then f sends $\frac{1}{2}$ to each incident 2-vertex;

(R4.2) If f is weak, then f sends $\frac{2}{3}$ to each incident 2-vertex and $\frac{1}{3n_3(f)}$ to each incident 3-vertex;

(R4.3) If f is good, then f sends $\frac{1}{6}$ to each special 3-vertex and to each incident 3-vertex that has at least one 7^+-neighbour. It then distributes its excess charge uniformly among other incident 3-vertices.

(R5) Every 6^+-face f sends $\frac{2}{3}$ to each incident 2-vertex and $\frac{1}{3}$ to each incident 3-vertex.

After carrying out (R0)-(R5), we denote by $\gamma(v)$ the excess charge of a 2-vertex v. Call a 2-vertex v *poor* if $\gamma(v) < 0$.

(R6) Let v be a 2-vertex incident to f_1 and f_2 with $\gamma(v) > 0$.

(R6.1) If f_1 and f_2 are both incident to poor 2-vertices, then v gives $\frac{\gamma(v)}{2}$ to each f_i and then f_i distributes $\frac{\gamma(v)}{2}$ uniformly among incident poor 2-vertices, where $i \in \{1, 2\}$;

(R6.2) If exactly one of f_1 and f_2 is incident to poor 2-vertices, say f_1, then v gives $\gamma(v)$ to f_1 and then f_1 distributes $\gamma(v)$ uniformly among incident poor 2-vertices.

Let $f \in F(G)$. First, we show that $\omega^*(f) \geqslant 0$. Since $g(G) \geqslant 5$, we see that $d(f) \geqslant 5$. If f is a 6^+-face, then $n_3(f) \leqslant d(f) - 2n_2(f)$ by Lemma 6.1. So by (R5) we have that $\omega^*(f) \geqslant d(f) - 4 - \frac{2}{3}n_2(f) - \frac{1}{3}n_3(f) \geqslant d(f) - 4 - \frac{2}{3}n_2(f) - \frac{1}{3}(d(f) - 2n_2(f)) = \frac{2}{3}d(f) - 4 \geqslant 0$. Now suppose that $d(f) = 5$. Then $\omega(f) = 1$. If f is bad, namely $n_2(f) = 2$, then we obtain that $\omega^*(f) \geqslant 1 - \frac{1}{2} \times 2 = 0$ by (R4.1). If f is weak, then $n_2(f) = 1$, and thus $\omega^*(f) \geqslant 1 - \frac{2}{3} - \frac{1}{3n_3(f)} \times n_3(f) = 0$ by (R4.2). If f is good, let $n_{3*}(f)$ denote the total number of the incident special 3-vertices and the incident

3-vertices that are adjacent to at least one 7^+-vertex. Then, by (R4.3), we conclude that $\omega^*(f) \geqslant 1 - \frac{1}{6}n_3(f) \geqslant 1 - \frac{1}{6} \times 5 > 0$.

Next, let $v \in V(G)$. We remind that G has minimum degree at least 2. If $d(v) \geqslant 10$, then $\omega^*(v) \geqslant d(v) - 4 - \frac{d(v)-4}{n_{3^-}(v)} \times n_{3^-}(v) = 0$ by (R3). Now suppose that $6 \leqslant d(v) \leqslant 9$. If $n_{4^+}(v) \geqslant 1$, then either $\omega^*(v) \geqslant d(v) - 4 - \frac{d(v)-4}{n_{3^-}(v)} \times n_{3^-}(v) = 0$ by (R2.1a) when v is a 6^7-vertex, or $\omega^*(v) \geqslant d(v) - 4 - \frac{1}{2}n_3(v) - \frac{d(v)-4-\frac{1}{2}n_3(v)}{n_2(v)} \times n_2(v) = 0$ by (R2.1b) when v is an 8^9-vertex. Otherwise, assume that $n_{4^+}(v) = 0$. By (R2.2), it is easy to deduce that $\omega^*(v) \geqslant d(v) - 4 - \frac{1}{6}n_{3^h}(v) - \frac{d(v)-4-\frac{1}{6}n_{3^h}(v)}{n_2(v)+n_{3^l}(v)} \times (n_2(v) + n_{3^l}(v)) = 0$. Noting that no 4-vertex participates in discharging argument, so $\omega^*(v) = \omega(v) = d(v) - 4 = 0$ for each 4-vertex v. If $d(v) = 5$, then $\omega(v) = 1$. One may easily obtain that $\omega^*(v) \geqslant 1 - \frac{1}{n_{3^-}(v)} \times n_{3^-}(v) = 0$ by (R1.1) if $n_{4^+}(v) \geqslant 1$ and $\omega^*(v) \geqslant 1 - 5 \times \frac{1}{5} = 0$ by (R1.2) otherwise.

What remains is to discuss the cases that $d(v) = 3$ and $d(v) = 2$. From the discharging rules we have the following two facts:

Fact 6.1. *Every 5^+-vertex sends a charge of at least $\frac{1}{6}$ to each heavy 3-neighbour by (R1.2), (R2) and (R3).*

Fact 6.2. *Each 7^+-vertex sends a charge of at least $\frac{1}{2}$ to each light 3-neighbour by (R2), (R3) and Lemma 6.7.*

Claim 6.1. *Each 3-vertex $v \in V(G)$ has a nonnegative final charge.*

Proof. Clearly, $\omega(v) = -1$. Let $N_G(v) = \{v_1, v_2, v_3\}$ and f_i denote the face incident to v by vv_i, vv_{i+1} as boundary edges, where indices are taken modulo 3. By Lemma 6.2, none of f_1, f_2, f_3 can be a bad face. If v is heavy, w.l.o.g., assume that v_1 and v_2 are 5^+-vertices. By Fact 6.1, $\tau(v_i \to v) \geqslant \frac{1}{6}$ for each $i \in \{1, 2\}$. Since $g(G) \geqslant 5$, we see that $d(f_i) \geqslant 5$ and thus $\tau(f_i \to v) \geqslant \frac{1}{6}$ for each $i \in \{1, 2, 3\}$ by (R4) and (R5). So if $d(v_3) \geqslant 5$, then we have that $\omega^*(v) \geqslant -1 + \frac{1}{6} \times 6 = 0$. Otherwise, assume $d(v_3) \leqslant 4$. If $d(f_1) \geqslant 6$, then, by (R5), $d(f_1 \to v) \geqslant \frac{1}{3}$. If $d(f_1) = 5$, then $n_3(f_1) \leqslant 3$ basing on the fact that v_1 and v_2 are both 5^+-vertices, and thus by (R4.2) and (R4.3), we calculate that $d(f_1 \to v) \geqslant \frac{1}{3}$. These two facts ensures us that v always gets a charge of at least $\frac{1}{3}$ from f_1. If either f_2 or f_3 is a 6^+-face, say f_2, then $\tau(f_2 \to v) \geqslant \frac{1}{3}$ by (R5), and, therefore, $\omega^*(v) \geqslant -1 + \frac{1}{3} + \frac{1}{3} + 3 \times \frac{1}{6} - \frac{1}{6} = 0$ by (R0). Otherwise, assume that $d(f_2) = d(f_3) = 5$. By definition, v is a special 3-vertex of f_2 and f_3, and hence $\omega^*(v) \geqslant -1 + \frac{1}{3} + 4 \times \frac{1}{6} = 0$.

Now suppose that v is a light 3-vertex. By Lemma 6.5, we may, w.l.o.g., assume that $d(v_1) \geqslant 5$ and both v_2 and v_3 are 4^--vertices. By Lemma 6.2, v cannot be incident to any bad 5-faces. It follows from (R4) and (R5) that v gets a charge of at least $\frac{1}{6}$ from each incident face. The following discussion is split into three cases depending on the degree of v_1.

Case 1. $d(v_1) \geqslant 7$. Then, by Fact 6.2, we have that $\tau(v_1 \to v) \geqslant \frac{1}{2}$, and, therefore, $\omega^*(v) \geqslant -1 + \frac{1}{2} + 3 \times \frac{1}{6} = 0$.

Case 2. $d(v_1) = 6$. If $n_{4^+}(v_1) \geqslant 1$, then $\tau(v_1 \to v) \geqslant \frac{2}{5}$ by (R2.1a). Otherwise, we know that $n_{3^h}(v_1) \geqslant 2$ by Lemma 6.7, and thus $\tau(v_1 \to v) \geqslant \frac{2-\frac{1}{6}\times 2}{4} = \frac{5}{12}$ by (R2.2). So if v is incident to at least one 6^+-face, then it sends $\frac{1}{3}$ to v by (R5), and here each of the other two faces incident to v still sends a charge of at least $\frac{1}{6}$ to v. It follows that $\omega^*(v) \geqslant -1 + \frac{2}{5} + \frac{1}{3} + 2 \times \frac{1}{6} = \frac{1}{15}$. Now suppose that $d(f_1) = d(f_2) = d(f_3) = 5$. Denote by $f_1 = [vv_1u_1u_2v_2]$, $f_2 = [vv_2u_3u_4v_3]$ and $f_3 = [vv_3u_5u_6v_1]$. Note that f_2 is good by Lemma 6.1.

- Fist assume that at least one of f_1 and f_3 is good. By symmetry, let f_1 be such a good 5-face. If $d(v_2) = 3$, then at least one of u_2, u_3 is a 5^+-vertex by Lemma 6.5, implying that $\tau(f_1 \to v) + \tau(f_2 \to v) \geqslant \min\{\frac{1}{3} + \frac{1}{5}, \frac{1}{4} + \frac{1}{4}\} = \frac{1}{2}$ by (R4.3). At this moment, $\tau(f_3 \to v) \geqslant \frac{1}{3\times 2} = \frac{1}{6}$ by (R4.2) if f_3 is weak, or $\tau(f_3 \to v) \geqslant \frac{1}{4}$ by (R4.3) if f_3 is good. Thus, $\omega^*(v) \geqslant -1 + \frac{2}{5} + \frac{1}{2} + \frac{1}{6} = \frac{1}{15}$. Now suppose that $d(v_2) = 4$. It is easy to derive that $\tau(f_1 \to v) \geqslant \frac{1}{2}$, $\tau(f_2 \to v) \geqslant \frac{1}{4}$ by (R4.3), and hence $\omega^*(v) \geqslant -1 + \frac{2}{5} + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{19}{60}$.
- Now assume that both f_1 and f_3 are weak. Then $d(u_1) = d(u_6) = 2$ and thus u_2, u_5 are both 7^+-vertices by Lemma 6.1. If at least one of v_2 and v_3 is 4-vertex, say v_2, then $\tau(f_1 \to v) \geqslant \frac{1}{3}$ by (R4.2), $\tau(f_2 \to v) \geqslant \frac{1}{4}$ by (R4.3), and $\tau(f_3 \to v) \geqslant \frac{1}{6}$ by (R4.2), which implies that $\omega^*(v) \geqslant -1 + \frac{2}{5} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} = \frac{3}{20}$. Otherwise, suppose that $d(v_2) = d(v_3) = 3$. By Lemma 6.11, we affirm that at least one of u_3, u_4 is a 5^+-vertex. So $\tau(f_2 \to v) \geqslant \frac{1-\frac{1}{6}\times 2}{2} = \frac{1}{3}$ by (R4.3), which implies that $\omega^*(v) \geqslant -1 + \frac{2}{5} + \frac{1}{3} + 2 \times \frac{1}{6} = \frac{1}{15}$.

Case 3. $d(v_1) = 5$. By Lemma 6.6, v_1 is adjacent to at least one 7^+-vertex. So $\tau(v_1 \to v) \geqslant \frac{1}{4}$ by (R1.1). If v is incident to at least two 6^+-faces, then $\omega^*(v) \geqslant -1 + 2 \times \frac{1}{3} + \frac{1}{4} + \frac{1}{6} = \frac{1}{12}$ by (R4) and (R5). Next, assume that v is incident to exactly one 6^+-face. By symmetry, we have two cases.

- $d(f_1) \geqslant 6$. Then $\tau(f_1 \to v) \geqslant \frac{1}{3}$ by (R5). Let $f_2 = [vv_2u_3u_4v_3]$ and $f_3 = [vv_3u_5u_6v_1]$. Recall that f_2 is good. So $\tau(f_2 \to v) \geqslant \frac{1}{5}$ by (R4.3). If f_3 is good, then $\tau(f_3 \to v) \geqslant \frac{1}{4}$ by (R4.3), and thus $\omega^*(v) \geqslant -1 + \frac{1}{4} + \frac{1}{3} + \frac{1}{5} + \frac{1}{4} = \frac{1}{30}$. Otherwise, assume that f_3 is weak. Then by Lemma 6.1, $d(u_6) = 2$ and $d(u_5) \geqslant 7$. If $d(v_3) = 4$, then $\tau(f_3 \to v) = \frac{1}{3}$ by (R4.2) and $\tau(f_2 \to v) \geqslant \frac{1}{4}$ by (R4.3). Therefore, $\omega^*(v) \geqslant -1 + \frac{1}{4} + \frac{1}{3} + \frac{1}{4} + \frac{1}{3} = \frac{1}{6}$. Otherwise, assume that $d(v_3) = 3$. By Lemma 6.8, v_3 must be a heavy 3-vertex, meaning that $d(u_4) \geqslant 5$. So $\tau(f_3 \to v) \geqslant \frac{1}{6}$ by (R4.2), $\tau(f_2 \to v) \geqslant \frac{1}{4}$ by (R4.3), and thus $\omega^*(v) \geqslant -1 + \frac{1}{4} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} = 0$.
- $d(f_2) \geqslant 6$. Again $\tau(f_2 \to v) \geqslant \frac{1}{3}$ by (R5). Let $f_1 = [vv_1u_1u_2v_2]$ and $f_3 = [vv_3u_5u_6v_1]$. If at least one of f_1 and f_3 is good, say f_1, then $\tau(f_1 \to v) \geqslant \frac{1}{4}$ by (R4.3), and $\tau(f_3 \to v) \geqslant \frac{1}{6}$ by (R4.2), thus $\omega^*(v) \geqslant -1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{3} + \frac{1}{6} = 0$. Next, suppose that f_1 and f_3 are both weak. Similarly, it is easy to deduce that $d(u_1) = d(u_6) = 2$ and thus u_2, u_5 are both 7^+-vertices by Lemma 6.1. If $d(v_i) = 4$ for some $i \in \{2, 3\}$, say v_2,

then by (R4.2), $\tau(f_1 \to v) \geqslant \frac{1}{3}$ and $\tau(f_3 \to v) \geqslant \frac{1}{6}$, and thus $\omega^*(v) \geqslant -1 + \frac{1}{4} + \frac{1}{3} + \frac{1}{3} + \frac{1}{6} = \frac{1}{12}$. Otherwise, suppose that $d(v_2) = d(v_3) = 3$. By Lemma 6.8, both v_2 and v_3 are heavy, and hence $\tau(v_2 \to v) = \tau(v_3 \to v) = \frac{1}{6}$ by (R0), implying that $\omega^*(v) \geqslant -1 + \frac{1}{4} + \frac{1}{6} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} \times 2 = \frac{1}{4}$.

Now, assume that $d(f_1) = d(f_2) = d(f_3) = 5$. Let $f_1 = [vv_1u_1u_2v_2]$, $f_2 = [vv_2u_3u_4v_3]$, $f_3 = [vv_3u_5u_6v_1]$, and let w_1, w_2 denote the other two neighbours of v_1 distinct to u_1, u_6 and v. Since $g(G) \geqslant 5$, neither w_1 nor w_2 can be incident to f_1, f_2 or f_3. Noting that $n_{4^+}(v_1) \geqslant 1$, we have to discuss following three subcases.

- $n_{4^+}(v_1) \geqslant 3$. Then $\tau(v_1 \to v) \geqslant \frac{1}{2}$ by (R1.1) and $\tau(f_i \to v) \geqslant \frac{1}{6}$ for each $i \in \{1, 2, 3\}$ by (R4.2) and (R4.3). Thus, $\omega^*(v) \geqslant -1 + \frac{1}{2} + 3 \times \frac{1}{6} = 0$.
- $n_{4^+}(v_1) = 2$. Then $\tau(v_1 \to v) \geqslant \frac{1}{3}$ by (R1.1). If $d(u_i) \geqslant 4$ for some fixed $i \in \{1, 6\}$, say $d(u_1) \geqslant 4$, then f_1 is good due to $d(u_2) \neq 2$ by Lemma 6.1, and thus $\tau(f_1 \to v) \geqslant \frac{1}{3}$ by (R4.3), $\tau(f_2 \to v) \geqslant \frac{1}{5}$ by (R4.3) since f_2 is good, implying that $\omega^*(v) \geqslant -1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} = \frac{1}{30}$. Now suppose that both u_1 and u_6 are 3^--vertices. Then $d(w_i) \geqslant 4$ for each $i \in \{1, 2\}$. If both f_1 and f_3 are good, then $\omega^*(v) \geqslant -1 + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{4} = \frac{1}{30}$. Otherwise, assume at least one of f_1 and f_3 is weak, say f_1. Then $d(u_1) = 2$, meaning that $d(u_2) \geqslant 7$ by Lemma 6.1.
  - If $d(v_2) = 4$, then $\tau(f_1 \to v) \geqslant \frac{1}{3}$, $\tau(f_2 \to v) \geqslant \frac{1}{4}$, $\tau(f_3 \to v) \geqslant \frac{1}{6}$ by (R4.2) and (R4.3), and hence $\omega^*(v) \geqslant -1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} = \frac{1}{12}$.
  - If $d(v_3) = 4$, then $\tau(f_1 \to v) \geqslant \frac{1}{6}$, $\tau(f_2 \to v) \geqslant \frac{1}{4}$, $\tau(f_3 \to v) \geqslant \frac{1}{3}$ by (R4.2) and (R4.3), and thus $\omega^*(v) \geqslant -1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{4} + \frac{1}{3} = \frac{1}{12}$.
  - Suppose that $d(v_2) = d(v_3) = 3$. By Lemma 6.8, we see that v_2 cannot be light. Namely, $d(u_3) \geqslant 5$. Since $d(u_5) \neq 2$, we know that $d(u_6) = 2$ if f_3 is weak, and thus $d(u_5) \geqslant 7$ by Lemma 6.1. Again, by Lemma 6.8, v_3 is heavy. That is, $d(u_4) \geqslant 5$, implying that $\tau(f_2 \to v) = \frac{1}{3}$ by (R4.3) and $\tau(f_i \to v) = \frac{1}{6}$ by (R4.2) for both $i \in \{1, 2\}$. Thus, $\omega^*(v) \geqslant -1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{3} + \frac{1}{6} = 0$. Otherwise, assume f_3 is good. Here, f_3 sends at least $\frac{1}{4}$ to v by (R4.3) and hence $\omega^*(v) \geqslant -1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{4} + \frac{1}{4} = 0$.
- $n_{4^+}(v_1) = 1$. Then $\tau(v_1 \to v) \geqslant \frac{1}{4}$ by (R1.1). By symmetry, we have three possibilities below:
  - f_1 and f_3 are both good. Lemma 6.5 ensures that $u_2, v_2, u_3, u_4, v_3, u_5$ cannot be all 3-vertices at the same time. That is, one of them must be a 4^+-vertex. If $d(u_2) \geqslant 4$, then $\tau(f_1 \to v) + \tau(f_2 \to v) + \tau(f_3 \to v) \geqslant \frac{1}{3} + \frac{1}{5} + \frac{1}{4} = \frac{47}{60}$. If $d(v_2) = 4$, then $\tau(f_1 \to v) + \tau(f_2 \to v) + \tau(f_3 \to v) \geqslant \frac{1}{3} + \frac{1}{4} + \frac{1}{4} = \frac{5}{6}$. If $d(u_3) \geqslant 4$, then $\tau(f_1 \to v) + \tau(f_2 \to v) + \tau(f_3 \to v) \geqslant \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$. Thus, in every case, one deduce that $\omega^*(v) \geqslant -1 + \frac{1}{4} + \frac{3}{4} = 0$.

- f_1 is weak and f_3 is good. By Lemma 6.1, $d(u_1) = 2$ and $d(u_2) \geqslant 7$. If $d(v_2) = 4$, then $\omega^*(v) \geqslant -1 + \frac{1}{4} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} = \frac{1}{12}$. If $d(v_3) = 4$, then $\omega^*(v) \geqslant -1 + \frac{1}{4} + \frac{1}{6} + \frac{1}{4} + \frac{1}{3} = 0$. Otherwise, assume that $d(v_2) = d(v_3) = 3$. By Lemma 6.8, we see that $d(u_3) \geqslant 5$. If $d(u_4) \geqslant 4$, then $\tau(f_2 \to v) \geqslant +\tau(f_3 \to v) \geqslant \frac{1-\frac{1}{6}}{2} + \frac{1}{4} = \frac{2}{3}$ by (R4.2) and (R4.3). Or else, we affirm that $d(u_5) \geqslant 5$ by Lemma 6.5, and thus $\tau(f_2 \to v) \geqslant +\tau(f_3 \to v) \geqslant \frac{1-\frac{1}{6}}{3} + \frac{1}{3} = \frac{11}{18}$ by (R4.2) and (R4.3). Consequently, $\omega^*(v) \geqslant -1 + \frac{1}{4} + \frac{1}{6} + \frac{11}{18} = \frac{1}{36}$.
- f_1 and f_3 are both weak. By Lemma 6.1, $d(u_1) = d(u_6) = 2$ and $d(u_i) \geqslant 7$ for each $i = 2, 5$. If $d(v_2) = 4$, then by (R4.2) and (R4.3), $\omega^*(v) \geqslant -1 + \frac{1}{4} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} = 0$. Otherwise, let $d(v_3) = d(v_4) = 3$. Again, Lemma 6.8 guarantees us that $d(u_3) \geqslant 5$ and $d(u_4) \geqslant 5$. By (R4.2) and (R4.3), $\tau(f_2 \to v) \geqslant 1 - \frac{1}{6} \times 2 = \frac{2}{3}$, and therefore $\omega^*(v) \geqslant -1 + \frac{1}{4} + \frac{1}{6} + \frac{2}{3} + \frac{1}{6} = \frac{1}{4}$. □

Before stating the next claim, using (R1)-(R3) and Lemma 6.6, we would like to present the following fact that concerns the charges given by a 5^+-vertex to each of its 2-neighbours.

Fact 6.3. *Let v be a k-vertex with a 2-neighbour u. We have the following:*
(1) $\tau(v \to u) \geqslant \frac{1}{4}$ if $k = 5$;
(2) $\tau(v \to u) \geqslant \frac{2}{5}$ if $k = 6$;
(3) $\tau(v \to u) \geqslant \frac{1}{2}$ if $k \in \{7, 8\}$;
(4) $\tau(v \to u) \geqslant \frac{3}{5}$ if $k \geqslant 9$.

Proof. (1) By Lemma 6.6, we have that $n_{4^+}(v) \geqslant 1$, and hence $\tau(v \to u) \geqslant \frac{1}{4}$ by (R1.1).

(2) We need to discuss two cases based on the value of $n_{4^+}(v)$. If $n_{4^+}(v) \geqslant 1$, then by (R2.1a), we have that $\tau(v \to u) \geqslant \frac{6-4}{5} = \frac{2}{5}$. Otherwise, assume that $n_{4^+}(v) = 0$. By Lemma 6.7, one may see that $n_{3^h}(v) \geqslant 2$. Further by (R2.2), we obtain that $\tau(v \to u) \geqslant \frac{6-4-2\times\frac{1}{6}}{4} = \frac{5}{12} > \frac{2}{5}$.

(3) First assume that $n_{4^+}(v) \geqslant 1$. If $k = 7$, then by (R2.1a), it is easy to deduce that $\tau(v \to u) \geqslant \frac{7-4}{6} = \frac{1}{2}$. If $k = 8$, then by (R2.1b), we have that $\tau(v \to u) \geqslant \frac{8-4}{7} = \frac{4}{7} > \frac{1}{2}$. Now assume that $n_{4^+}(v) = 0$. By Lemma 6.7, $n_{3^h}(v) \geqslant 2$. Thus, by applying (R2.2), $\tau(v \to u) \geqslant \frac{7-4-2\times\frac{1}{6}}{5} = \frac{8}{15} > \frac{1}{2}$ when $k = 7$, and $\tau(v \to u) \geqslant \frac{8-4-2\times\frac{1}{6}}{6} = \frac{11}{18} > \frac{1}{2}$ when $k = 8$.

(4) First suppose that $k = 9$. If $n_{4^+}(v) \geqslant 1$, then by (R2.1b), we can deduce that $\tau(v \to u) \geqslant \frac{9-4}{8} = \frac{5}{8} > \frac{3}{5}$. Otherwise, assume that $n_{4^+}(v) = 0$. By (R2.2), $\tau(v \to u)\frac{9-4-2\times\frac{1}{6}}{7} = \frac{2}{3} > \frac{3}{5}$. Now suppose that $k \geqslant 10$. By (R3), we may conclude that $\tau(v \to u) \geqslant \frac{10-4}{10} = \frac{3}{5}$. □

Recall that give a face f and an edge of f, f_e denotes the face that is adjacent to f by the common edge e.

Claim 6.2. *Each 2-vertex v has nonnegative final charge.*

Proof. Let v_1, v_2 denote the neighbours of v. Clearly, $\omega(v) = -2$. By Lemma 6.1, w.l.o.g., suppose that $d(v_1) \geqslant 5$ and $d(v_2) \geqslant 7$. By Fact 6.3, we see that $\tau(v_1 \to v) \geqslant \frac{1}{4}$ and $\tau(v_2 \to v) \geqslant \frac{1}{2}$. If v is not incident to any bad 5-face, then each incident face sends a charge of at least $\frac{2}{3}$ to v by (R4.2) and (R5), implying that $\omega^*(v) \geqslant -2+\frac{1}{4}+\frac{1}{2}+2\times\frac{2}{3} = \frac{1}{12}$.

Next, suppose first that v is incident to exactly one bad 5-face. Denote by $f = [vv_2v_3v_4v_1]$. Note that v gets a total charge of at least $\frac{1}{2} + \frac{1}{2} + \frac{2}{3} = \frac{5}{3}$ from v_2 and its incident faces by (R4) and (R5). It means that in next discussion, we only need to show that $\tau(v_1 \to v) \geqslant \frac{1}{3}$, that would imply $\omega^*(v) \geqslant -2 + \frac{5}{3} + \frac{1}{3} = 0$. If $d(v_1) \geqslant 6$, then we are done since $\tau(v_1 \to v) \geqslant \frac{2}{5} > \frac{1}{3}$ by Fact 6.3 (2)-(4). Now let $d(v_1) = 5$. By Lemma 6.6, $n_{4^+}(v_1) \geqslant 1$. By (R1.1), we may further assume that $n_{4^+}(v_1) = 1$. Actually, observe that the unique 4^+-neighbour is a 7^+-vertex by Lemma 6.6. At this moment, if $d(v_2) \geqslant 9$, then $\tau(v_2 \to v) \geqslant \frac{3}{5}$ by Fact 6.3 (4), and hence $\omega^*(v) \geqslant -2 + \frac{1}{4} + \frac{3}{5} + \frac{1}{2} + \frac{2}{3} = \frac{1}{60}$. If $d(v_2) = 7$, then by Lemma 6.9, we have $n_{4^+}(v_2) \geqslant 2$, meaning that $\tau(v_2 \to v) \geqslant \frac{3}{5}$ by (R2.1a) and similarly that $\omega^*(v) \geqslant -2 + \frac{1}{4} + \frac{3}{5} + \frac{1}{2} + \frac{2}{3} = \frac{1}{60}$. Now suppose that $d(v_2) = 8$. If $n_{4^+}(v_2) \geqslant 2$, then we have that $\tau(v_2 \to v) \geqslant \frac{2}{3}$ by (R.2.1b) and thus $\omega^*(v) \geqslant -2 + \frac{1}{4} + \frac{2}{3} + \frac{1}{2} + \frac{2}{3} = \frac{1}{12}$. If $n_{4^+}(v_2) = 0$, then by Lemma 6.7, v_2 has at least two heavy 3-neighbours. So by (R2.2), we have that $\tau(v_2 \to v) \geqslant \frac{4-\frac{1}{6}\times 2}{6} = \frac{11}{18}$. Thus, $\omega^*(v) \geqslant -2 + \frac{1}{4} + \frac{11}{18} + \frac{1}{2} + \frac{2}{3} = \frac{1}{36}$. In what follows, it suffice to discuss the case that $n_{4^+}(v_2) = 1$.

- If $d(v_4) = 2$, then $d(v_3) \geqslant 7$ by Lemma 6.1. Moreover, $n_3(v_2) \geqslant 1$ by Lemma 6.10. So $\tau(v_2 \to v) \geqslant \frac{4-\frac{1}{2}}{6} = \frac{7}{12}$ by (R2.1b), implying that $\omega^*(v) \geqslant -2+\frac{1}{4}+\frac{1}{2}+\frac{7}{12}+\frac{2}{3} = 0$.
- If $d(v_3) = 2$, then $d(v_4) \geqslant 7$ due to $n_{7^+}(v_1) = 1$. By (R2.1b), $\tau(v_2 \to v) \geqslant \frac{4}{7}$. Now we look at $f_{v_2v_3}$. By (R4.1), (R4.2) and (R5), $\tau(f_{v_2v_3} \to v) \geqslant \frac{1}{2}$ if $f_{v_2v_3}$ is bad or $\tau(f_{v_2v_3} \to v) \geqslant \frac{2}{3}$ otherwise. It follows that $\omega^*(v_3) \geqslant -2 + \frac{1}{2} + \frac{4}{7} + \frac{1}{2} + \frac{1}{2} = \frac{1}{14}$. By (R6), v gets a charge of at least $\frac{1}{14} \times 2 = \frac{1}{28}$ from v_3. Therefore, $\omega^*(v) \geqslant -2 + \frac{1}{4} + \frac{1}{2} + \frac{4}{7} + \frac{2}{3} + \frac{1}{28} = \frac{1}{42}$.

Now suppose that v is incident to two bad 5-faces $f = [vv_1u_1u_2v_2]$ and $g = [v_1vv_2u_3u_4]$. By (R4.1), $\tau(f \to v) = \tau(g \to v) = \frac{1}{2}$. If both v_1 and v_2 have degree at least 7, then by Fact 6.3 (3)-(4), we have that $\omega^*(v) \geqslant -2 + 4 \times \frac{1}{2} = 0$. Next, by Lemma 6.1 we can assume that $5 \leqslant d(v_1) \leqslant 6$ and $d(v_2) \geqslant 7$. It suffices to handle three cases by the situation of incident 2-vertices.

Case 1: $d(u_1) = d(u_4) = 2$.

By Lemma 6.1, u_2 and u_3 are 7^+-vertices. By (R2.1) and (R3), $\tau(v_2 \to v) \geqslant \frac{3}{5}$. If $d(v_1) = 6$, then $\tau(v_1 \to v) \geqslant \frac{2}{5}$ by Fact 6.3 (2), and thus $\omega^*(v) \geqslant -2 + \frac{2}{5} + \frac{3}{5} + \frac{1}{2} + \frac{1}{2} = 0$. Next, suppose that $d(v_1) = 5$. Then $n_{7^+}(v_1) \geqslant 1$ by Lemma 6.6. Let w_1, w_2 denote the other two neighbours of v_1 and assume that $d(w_1) \geqslant 7$.

- $n_{4^+}(v_2) \geqslant 3$. Then $\tau(v_2 \to v) \geqslant \frac{3}{4}$ by (R2.1) and (R3). Hence, $\omega^*(v) \geqslant -2 + \frac{1}{4} + \frac{3}{4} + \frac{1}{2} + \frac{1}{2} = 0$ by (R1.1).

- $n_{4^+}(v_2) = 2$. By Lemma 6.12 (1), $d(v_2) \geqslant 8$. If $d(v_2) = 8$, then by Lemma 6.13 (1), v_2 has at least two 3-neighbours. So $\tau(v_2 \to v) \geqslant \frac{4-\frac{1}{2}\times 2}{8-4} = \frac{3}{4}$ by (R2.1b). If $d(v_2) = 9$, then by Lemma 6.13 (2), v_2 has at least one 3-neighbour, and thus $\tau(v_2 \to v) \geqslant \frac{5-\frac{1}{2}}{9-3} = \frac{3}{4}$ by (R2.1b). If $d(v_2) \geqslant 10$, then $\tau(v_2 \to v) = \frac{d(v_2)-4}{n_{3^-}(v_2)} \geqslant 1 - \frac{2}{d(v_2)-2} \geqslant \frac{3}{4}$ by (R3). Hence, in each case, one may always deduce that $\omega^*(v) \geqslant -2 + \frac{1}{4} + \frac{3}{4} + \frac{1}{2} + \frac{1}{2} = 0$.

Case 2: $d(u_2) = d(u_3) = 2$.

By Lemma 6.1, u_1 and u_4 are both 5^+-vertices, and so $\tau(v_1 \to v) \geqslant \frac{1}{3}$ by (R1.1) and $\tau(v_2 \to v) \geqslant \frac{1}{2}$ by Fact 6.3 (3)-(4). If $d(v_1) = 6$, then $\tau(v_1 \to v) \geqslant \frac{1}{2}$ by (R2.1a), meaning that $\omega^*(v) \geqslant -2 + 4 \times \frac{1}{2} = 0$. Next, suppose that $d(v_1) = 5$. Let w_1, w_2 denote the other two neighbours of v_1. If one of w_1 and w_2 is a 4^+-vertex, then $\tau(v_1 \to v) \geqslant \frac{1}{2}$ by (R1.1) and thus $\omega^*(v) \geqslant -2 + 4 \times \frac{1}{2} = 0$. Now suppose that both w_1 and w_2 are 3^--vertices. By Lemma 6.14 (1), at least one of u_1 and u_4 is of degree at least 8. W.l.o.g., assume that $d(u_1) \geqslant 8$. Moreover, by Lemma 6.12 (2), $d(v_2) \geqslant 8$. If $n_{4^+}(v_2) \geqslant 2$, then $\tau(v_2 \to v) \geqslant \frac{4}{8-2} = \frac{2}{3}$ by (R2.1b). So $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{2}{3} + \frac{1}{2} + \frac{1}{2} = 0$. Next, consider the case that $n_{4^+}(v_2) \leqslant 1$.

Case 2a: $d(v_2) = 8$. If $n_{4^+}(v_2) = 0$, then v_2 is adjacent to at least two heavy 3-vertices by Lemma 6.7, and thus $\tau(v_2 \to v) \geqslant \frac{4-\frac{1}{6}\times 2}{8-2} = \frac{11}{18}$ by (R2.2). If $n_{4^+}(v_2) = 1$, it is easy to obtain that $\tau(v_2 \to v) \geqslant \frac{4}{7}$ by (R2.1a). So both above possibilities guarantee us that $\tau(v_2 \to v) \geqslant \min\{\frac{11}{18}, \frac{4}{7}\} = \frac{4}{7}$.

Case 2a(1): If $f_{u_2v_2}$ is not bad, then $\tau(f_{u_2v_2} \to u_2) \geqslant \frac{2}{3}$ by (R4.2), (R4.3) and (R5), and thus $\omega^*(u_2) \geqslant -2 + \frac{1}{2} + \frac{4}{7} + \frac{1}{2} + \frac{2}{3} = \frac{5}{21}$. So $\tau(u_2 \to v) \geqslant \frac{5}{21}$ by (R6.2), implying that $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{4}{7} + \frac{1}{2} + \frac{1}{2} + \frac{5}{21} = \frac{1}{7}$.

Case 2a(2): Now assume $f_{u_2v_2}$ is bad. Namely, $f_{u_2v_2}$ is a 5-face incident to exactly two 2-vertices. Let $f_{u_2v_2} = [u_1u_2v_2xy]$. If $d(x) = 2$, then $d(y) \geqslant 5$ by Lemma 6.1 and thus $n_{4^+}(u_1) \geqslant 2$. So $\tau(u_1 \to u_2) \geqslant \frac{5}{8}$ by (R2.1b) and (R5), implying that $\omega^*(u_2) \geqslant -2+\frac{5}{8}+\frac{4}{7}+\frac{1}{2}\times 2 = \frac{11}{56}$. By (R6), $\tau(u_2 \to v) \geqslant \frac{11}{112}$. Therefore, $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{4}{7} + \frac{1}{2} \times 2 + \frac{11}{112} = \frac{1}{336}$. Otherwise, assume that $d(y) = 2$. By Lemma 6.1, we assert that $d(x) \geqslant 5$.

- If $d(u_1) \geqslant 9$, then $\tau(u_1 \to u_2) \geqslant \frac{5}{8}$ by (R2.1b) and (R3), and similarly we obtain that $\omega^*(u_2) \geqslant -2 + \frac{5}{8} + \frac{4}{7} + \frac{1}{2} \times 2 = \frac{11}{56}$, and, therefore, $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{4}{7} + \frac{1}{2} \times 2 + \frac{11}{112} = \frac{1}{336}$ by (R6).
- $d(u_1) = 8$. If $d(u_4) \leqslant 7$, then $n_{4^+}(u_1) \geqslant 2$ by Lemma 6.14 (2) and thus $\tau(u_1 \to u_2) \geqslant \frac{5}{8}$. By the same calculation as above, we have that $\omega^*(v) \geqslant \frac{1}{336}$. So next suppose that $d(u_4) \geqslant 8$. Now we look at the face $f_{u_3v_2}$. By a similar discussion as that of Case 2a(1), we may suppose that $f_{u_3v_2}$ is bad. Let $f_{u_3v_2} = [u_4u_3v_2zw]$. As $n_{4^+}(v_2) \leqslant 1$, we are sure that $d(z) = 2$, which implies that $d(w) \geqslant 5$ by Lemma 6.1. Similarly, $\tau(u_4 \to u_3) \geqslant \frac{5}{8}$. So $\omega^*(u_3) \geqslant -2 + \frac{5}{8} + \frac{4}{7} + \frac{1}{2} \times 2 = \frac{11}{56}$. By (R6), $\tau(u_3 \to v) \geqslant \frac{11}{112}$. Consequently, $\omega^*(v) \geqslant -2+\frac{1}{3}+\frac{4}{7}+\frac{1}{2}\times 2+\frac{11}{112} = \frac{1}{336}$.

Case 2b: $d(v_2) = 9$. If $n_{4^+}(v_2) = 0$, then by Lemma 6.7, we have that $n_{3^h}(v_2) \geqslant 2$, and so $\tau(v_2 \to v) \geqslant \frac{5-2\times\frac{1}{6}}{9-2} = \frac{2}{3}$ by (R2.2) and thus $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{2}{3} + \frac{1}{2} \times 2 = 0$. Now suppose that $n_{4^+}(v_2) = 1$. By (R2.1b), $\tau(v_2 \to v) \geqslant \frac{5}{8}$. One may further assume that $d(u_1) \geqslant 8$ by Lemma 6.14 (1). As $n_{4^+}(u_1) \geqslant 1$, by (R2.1b) and (R3), we see that $\tau(u_1 \to u_2) \geqslant \frac{4}{7}$. If $f_{u_2v_2}$ is not bad, then $\tau(f_{u_2v_2} \to u_2) \geqslant \frac{2}{3}$. Else, $\tau(f_{u_2v_2} \to u_2) \geqslant \frac{1}{2}$. Thus $\omega^*(u_2) \geqslant -2 + \frac{4}{7} + \frac{5}{8} + \frac{1}{2} \times 2 = \frac{11}{56}$. By (R6), $\tau(u_2 \to v) \geqslant \frac{11}{112}$ and hence $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{5}{8} + \frac{1}{2} \times 2 + \frac{11}{112} = \frac{19}{336}$.

Case 2c: $d(v_2) \geqslant 10$. Then $\tau(v_2 \to v) \geqslant \frac{3}{5}$ by Fact 6.3 (4). One may deduce that $\omega^*(u_2) \geqslant -2 + \frac{4}{7} + \frac{3}{5} + \frac{1}{2} \times 2 = \frac{6}{35}$, and thus $\tau(u_2 \to v) \geqslant \frac{3}{35}$ by (R6). We conclude that $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{3}{5} + \frac{1}{2} \times 2 + \frac{3}{35} = \frac{14}{105}$.

Case 3: $d(u_2) = d(u_4) = 2$.

Then $d(u_1) \geqslant 5$ and $d(u_3) \geqslant 7$ by Lemma 6.1. We have two subcases depending on $d(v_1)$.

Case 3a: $d(v_1) = 6$. Then $\tau(v_1 \to v) \geqslant \frac{2}{5}$ by Fact 6.3 (2) and $\tau(v_2 \to v) \geqslant \frac{1}{2}$ by Fact 6.3 (3)-(4). If $n_{4^+}(v_1) \geqslant 2$ or $n_{4^+}(v_2) \geqslant 2$, then either $\tau(v_1 \to v) \geqslant \frac{1}{2}$ or $\tau(v_2 \to v) \geqslant \frac{3}{5}$ by (R2.1a), (R2.1b) and (R3). We obtain that $\omega^*(v) \geqslant -2 + \frac{1}{2} \times 4 = 0$ or $\omega^*(v) \geqslant -2 + \frac{2}{5} + \frac{3}{5} + \frac{1}{2} \times 2 = 0$. Next, suppose that $n_{4^+}(v_1) = n_{4^+}(v_2) = 1$. By Lemma 6.19 (1), $d(v_2) \geqslant 8$. If $d(v_2) \geqslant 9$, then $\tau(v_2 \to v) \geqslant \frac{3}{5}$ by Fact 6.3 (4), implying that $\omega^*(v) \geqslant -2 + \frac{2}{5} + \frac{3}{5} + \frac{1}{2} \times 2 = 0$. In what follows, we suppose $d(v_2) = 8$. Then $\tau(v_2 \to v) \geqslant \frac{4}{7}$ by (R2.1b).

- First suppose that $f_{u_2v_2}$ is not bad. If $d(u_1) = 5$, then u_1 has at least one 7^+-neighbour different from v_1 by Lemma 6.6. This means that $\tau(u_1 \to u_2) \geqslant \frac{1}{3}$ by (R1.1). If $d(u_1) \geqslant 6$, then $\tau(u_1 \to u_2) \geqslant \frac{2}{5}$ by Fact 6.3 (2)-(4). So in each case, u_1 sends at least $\frac{1}{3}$ to u_2. Thus, $\omega^*(u_2) \geqslant -2 + \frac{1}{3} + \frac{4}{7} + \frac{1}{2} + \frac{2}{3} = \frac{1}{14}$. By (R6.1), $\tau(u_2 \to v) \geqslant \frac{1}{14}$. Therefore, $\omega^*(v) \geqslant -2 + \frac{2}{5} + \frac{4}{7} + \frac{1}{2} \times 2 + \frac{1}{14} = \frac{3}{70}$.
- Now suppose that $f_{u_2v_2}$ is a bad 5-face. Let $f_{u_2v_2} = [u_1u_2v_2xy]$. As $n_{4^+}(v_2) = 1$, we affirm that $d(x) = 2$, which leads to $d(y) \geqslant 5$ by Lemma 6.1. If $d(u_1) \geqslant 6$, then by (R2.1) and (R3), we have that $\tau(u_1 \to u_2) \geqslant \frac{1}{2}$. If $d(u_1) = 5$, then by Lemma 6.19 (2), $n_{4^+}(u_1) \geqslant 3$. So $\tau(u_1 \to u_2) \geqslant \frac{1}{2}$ by (R1.1). In both cases, we always derive that $\omega^*(u_2) \geqslant -2 + \frac{1}{2} + \frac{4}{7} + \frac{1}{2} \times 2 = \frac{1}{14}$. By (R6), $\tau(u_2 \to v) \geqslant \frac{1}{28}$, implying that $\omega^*(v) \geqslant -2 + \frac{2}{5} + \frac{4}{7} + \frac{1}{2} \times 2 + \frac{1}{28} = \frac{1}{140}$.

Case 3b: $d(v_1) = 5$. By (R1.1), $\tau(v_1 \to v) \geqslant \frac{1}{4}$. By Fact 6.3 (3)-(4), $\tau(v_2 \to v) \geqslant \frac{1}{2}$. Let w_1, w_2 denote the other two neighbours of v_1. If $n_{4^+}(v_1) \geqslant 3$ or $n_{4^+}(v_2) \geqslant 3$, then $\tau(v_1 \to v) \geqslant \frac{1}{2}$ by (R1.1) or $\tau(v_2 \to v) \geqslant \frac{3}{4}$ by (R2.1b) and (R3). Thus, $\omega^*(v) \geqslant -2 + \frac{1}{2} \times 4 = 0$ or $\omega^*(v) \geqslant -2 + \frac{1}{4} + \frac{3}{4} + \frac{1}{2} \times 2 = 0$. Next, assume that $n_{4^+}(v_1) \leqslant 2$ and $n_{4^+}(v_2) \leqslant 2$.

Case 3b(1): $d(v_2) = 7$. Let $x_1, \ldots, x_4$ denote the other four neighbours of v_2 different from u_2, u_3 and v. By Lemma 6.15, we assert that $n_{4^+}(v_1) = n_{4^+}(v_2) = 2$. Then, $\tau(v_1 \to v) \geqslant \frac{1}{3}$ by (R1.1) and $\tau(v_2 \to v) \geqslant \frac{3}{5}$ by (R2.1a).

- Suppose that $f_{u_2v_2}$ is not bad. If $d(u_1) = 5$, then by Lemma 6.6, $n_{7^+}(u_1) \geqslant 1$ and thus $n_{4^+}(u_1) \geqslant 2$. By (R1.1), $\tau(u_1 \to u_2) \geqslant \frac{1}{3}$. If $d(u_1) \geqslant 6$, then $\tau(u_1 \to u_2) \geqslant \frac{2}{5}$ by Fact 6.3 (2)-(4). So u_2 gets a charge of at least $\frac{1}{3}$ from u_1. Therefore, $\omega^*(u_2) \geqslant -2 + \frac{1}{3} + \frac{3}{5} + \frac{1}{2} + \frac{2}{3} = \frac{1}{10}$. By (R6), $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{3}{5} + \frac{1}{2} \times 2 + \frac{1}{10} = \frac{1}{30}$.
- Suppose that $f_{u_2v_2}$ is a bad 5-face. Let $f_{u_2v_2} = [u_1u_2v_2x_1y]$.
  - $d(u_1) = 5$. By Lemma 6.16 (1), we are sure that $d(y) \neq 2$, which implies that $d(x_1) = 2$ and $d(y) \geqslant 5$ by Lemma 6.1. By Lemma 6.16 (2.1), $n_{4^+}(u_1) \geqslant 3$. By (R1.1), $\tau(u_1 \to u_2) \geqslant \frac{1}{2}$. Thus, $\omega^*(u_2) \geqslant -2 + \frac{1}{2} + \frac{3}{5} + \frac{1}{2} \times 2 = \frac{1}{10}$. Next, we show that x_1 is not a poor 2-vertex, which ensures us that u_2 sends all of its extra charge to v by (R6). Let f' denote the face adjacent to $f_{u_2v_2}$ by common edges yx_1 and x_1v_2. If f' is not bad, then $\omega^*(x_1) \geqslant -2 + \frac{1}{4} + \frac{3}{5} + \frac{1}{2} + \frac{2}{3} = \frac{1}{60}$. Otherwise, assume that f' is a bad 5-face. If $d(y) = 5$, then by Lemma 6.16 (2.2), we know that $n_{4^+}(y) \geqslant 3$ and then $\tau(y \to x_1) \geqslant \frac{1}{2}$ by (R1.1). If $d(y) \geqslant 6$, then $\tau(y \to x_1) \geqslant \frac{2}{5}$ by Fact 6.3 (2)-(4). So x_1 always gets a charge of at least $\frac{2}{5}$ from y, implying that $\omega^*(x_1) \geqslant -2 + \frac{2}{5} + \frac{3}{5} + \frac{1}{2} \times 2 = 0$. Therefore, $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{3}{5} + \frac{1}{2} \times 2 + \frac{1}{10} = \frac{1}{30}$.
  - $d(u_1) = 6$. There are two possible cases below:
    * $d(y) = 2$. Then $d(x_1) \geqslant 7$ by Lemma 6.1. By Lemma 6.16 (1), we see that $n_{4^+}(u_1) \geqslant 2$, meaning that $\tau(u_1 \to u_2) \geqslant \frac{1}{2}$ by (R2.1a). It follows that $\omega^*(u_2) \geqslant -2 + \frac{1}{2} + \frac{3}{5} + \frac{1}{2} \times 2 = \frac{1}{10}$. By Fact 6.3 (3)-(4), y gets a charge of at least $\frac{1}{2}$ from x_1. Therefore, $\omega^*(y) \geqslant -2 + \frac{1}{2} \times 4 = 0$. This means that u_2 sends all its extra charge of $\frac{1}{10}$ to v. Hence, $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{3}{5} + \frac{1}{2} \times 2 + \frac{1}{10} = \frac{1}{30}$.
    * $d(x_1) = 2$. Then $d(y) \geqslant 5$, implying that $n_{4^+}(u_1) \geqslant 2$. If $n_{4^+}(u_1) \geqslant 3$, then $\tau(u_1 \to u_2) \geqslant \frac{2}{3}$ by (R2.1a) and thus $\omega^*(u_2) \geqslant -2 + \frac{2}{3} + \frac{3}{5} + \frac{1}{2} \times 2 = \frac{4}{15}$. By (R6), $\tau(u_2 \to v) \geqslant \frac{2}{15}$ and so $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{3}{5} + \frac{1}{2} \times 2 + \frac{2}{15} = \frac{1}{15}$. Now assume $n_{4^+}(u_1) = 2$. Then $\tau(u_1 \to u_2) \geqslant \frac{1}{2}$ by (R2.1a) and so $\omega^*(u_2) \geqslant -2 + \frac{1}{2} + \frac{3}{5} + \frac{1}{2} \times 2 = \frac{1}{10}$. Similarly, we have to show that u_2 does not need to send any charge to x_1. Let f' denote the face adjacent to $f_{u_2v_2}$ by common edges yx_1 and x_1v_2. If f' is not bad, then $\omega^*(x_1) \geqslant -2 + \frac{1}{4} + \frac{3}{5} + \frac{1}{2} + \frac{2}{3} = \frac{1}{60}$. Otherwise, assume that f' is a bad 5-face. If $d(y) \geqslant 6$, then $\omega^*(x_1) \geqslant -2 + \frac{2}{5} + \frac{3}{5} + \frac{1}{2} \times 2 = 0$. Else, $d(y) = 5$. By Lemma 6.16 (2.2), $n_{4^+}(y) \geqslant 3$, and thus $\tau(y \to x_1) \geqslant \frac{1}{2}$, implying that $\omega^*(x_1) \geqslant -2 + \frac{1}{2} + \frac{3}{5} + \frac{1}{2} \times 2 = \frac{1}{10}$. Consequently, u_2 sends all its extra charge of $\frac{1}{10}$ to v by (R6.2), and, therefore, $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{3}{5} + \frac{1}{2} \times 2 + \frac{1}{10} = \frac{1}{30}$.

- $d(u_1) \geqslant 7$. If $d(x_1) = 2$, then $d(y) \geqslant 5$ by Lemma 6.1, and thus $\tau(u_1 \to u_2) \geqslant \frac{3}{5}$. So $\omega^*(u_2) \geqslant -2 + \frac{3}{5} \times 2 + \frac{1}{2} \times 2 = \frac{1}{5}$. By (R6), $\tau(u_2 \to v) \geqslant \frac{1}{10}$. So $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{3}{5} + \frac{1}{2} \times 2 + \frac{1}{10} = \frac{1}{30}$. Otherwise, assume that $d(y) = 2$. Then $\omega^*(u_2) \geqslant -2 + \frac{1}{2} + \frac{3}{5} + \frac{1}{2} \times 2 = \frac{1}{10}$. By Lemma 6.16 (1), $d(x_1) \geqslant 6$. If $d(x_1) \geqslant 7$, then $\tau(x_1 \to y) \geqslant \frac{1}{2}$ by Fact 6.3 (3)-(4). If $d(x_1) = 6$, then $n_{4^+}(x_1) \geqslant 2$ by Lemma 6.16 (1). So $\tau(x_1 \to y) \geqslant \frac{1}{2}$ by (R2.1a). Thus, $\omega^*(y) \geqslant -2 + \frac{1}{2} \times 4 = 0$. Hence, by (R6.2), u_2 sends $\frac{1}{10}$ to v directly, and, therefore, $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{3}{5} + \frac{1}{2} \times 2 + \frac{1}{10} = \frac{1}{30}$.

Case 3b(2): $d(v_2) = 8$. Let $x_1, \ldots, x_5$ denote the other five neighbours of v_2. Recall that $n_{4^+}(v_1) \leqslant 2$ and $n_{4^+}(v_2) \leqslant 2$. By symmetry, we have three cases:

- $n_{4^+}(v_1) = n_{4^+}(v_2) = 2$. By (R1.1), $\tau(v_1 \to v) \geqslant \frac{1}{3}$. By (R2.1a), $\tau(v_2 \to v) \geqslant \frac{2}{3}$. Thus, $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{2}{3} + \frac{1}{2} \times 2 = 0$.
- $n_{4^+}(v_1) = 2$ and $n_{4^+}(v_2) = 1$. Namely, $x_1, \ldots, x_5$ are all 3^--vertices. By Lemma 6.17 (2.1), we have $n_3(v_2) \geqslant 2$, implying that $\tau(v_2 \to v) \geqslant \frac{4-\frac{1}{2}\times 2}{8-3} = \frac{3}{5}$ by (R2.1b).
  - Assume $f_{u_2v_2}$ is not bad. By Lemma 6.6, u_1 has a 7^+-neighbour if $d(u_1) = 5$. So $\tau(u_1 \to u_2) \geqslant \frac{1}{3}$. Thus, $\omega^*(u_2) \geqslant -2 + \frac{1}{3} + \frac{3}{5} + \frac{1}{2} + \frac{2}{3} = \frac{1}{10}$, and so $\tau(u_2 \to v) \geqslant \frac{1}{10}$ by (R6.2). Hence, $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{3}{5} + \frac{1}{2} \times 2 + \frac{1}{10} = \frac{1}{30}$.
  - Assume $f_{u_2v_2}$ is a bad 5-face. Let $f_{v_2u_2} = [u_1u_2v_2x_1y]$. Then by Lemma 6.1, $d(x_1) = 2$ and $d(y) \geqslant 5$. This ensures us that $n_{4^+}(u_1) \geqslant 2$.
    * $d(u_1) = 5$. By Lemma 6.17 (2.2.1), $n_{4^+}(u_1) \geqslant 3$, implying that $\tau(u_1 \to u_2) \geqslant \frac{1}{2}$ by (R1.1). Thus we have that $\omega^*(u_2) \geqslant -2+\frac{1}{2}+\frac{3}{5}+\frac{1}{2}\times 2 = \frac{1}{10}$. Similarly, we show that x_1 is not poor and thus u_2 sends all its extra charge to v. If $d(y) \geqslant 6$, then $\tau(y \to x_1) \geqslant \frac{1}{2}$ by (R2.1a). Thus, $\omega^*(x_1) \geqslant -2 + \frac{1}{2} + \frac{3}{5} + \frac{1}{2} \times 2 = \frac{1}{10}$. Now assume $d(y) = 5$. Let f' denote the face adjacent to $f_{u_2v_2}$ by common edges yx_1 and x_1v_2. If f' is not bad, then it sends a charge of $\frac{2}{3}$ to x_1, and, therefore, $\omega^*(x_1) \geqslant -2 + \frac{1}{4} + \frac{3}{5} + \frac{1}{2} + \frac{2}{3} = \frac{1}{60}$. Otherwise, assume f' is a bad 5-face. By Lemma 6.17 (2.2.2), we see that $n_{4^+}(y) \geqslant 3$, so $\tau(y \to x_1) \geqslant \frac{1}{2}$ by (R1.1). Thus we have that $\omega^*(x_1) \geqslant -2 + \frac{1}{2} + \frac{3}{5} + \frac{1}{2} \times 2 = \frac{1}{10}$. Consequently, $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{3}{5} + \frac{1}{2} \times 2 + \frac{1}{10} = \frac{1}{30}$.
    * $d(u_1) = 6$. If $n_{4^+}(u_1) \geqslant 3$, then $\tau(u_1 \to u_2) \geqslant \frac{2}{3}$ by (R2.1a) and thus $\omega^*(u_2) \geqslant -2+\frac{2}{3}+\frac{3}{5}+\frac{1}{2}\times 2 = \frac{4}{15}$. By (R6), $\tau(u_2 \to v) \geqslant \frac{2}{15}$. So $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{3}{5} + \frac{1}{2} \times 2 + \frac{2}{15} = \frac{1}{15}$. Now suppose $n_{4^+}(u_1) = 2$. Then $\tau(u_1 \to u_2) \geqslant \frac{1}{2}$ by (R2.1a). Similarly, if $f_{x_1v_2}$ is not bad, then $\omega^*(x_1) \geqslant -2 + \frac{1}{4} + \frac{3}{5} + \frac{1}{2} + \frac{2}{3} = \frac{1}{60}$. Else, $f_{x_1v_2}$ is bad. If $d(y) \geqslant 6$, then $\tau(y \to x_1) \geqslant \frac{2}{5}$. If $d(y) = 5$, then by Lemma 6.17 (2.2.2), it has at least three

4^+-neighbours. So in each case, $\tau(y \to x_1) \geqslant \frac{1}{2}$ by (R1.1). Hence, $\omega^*(x_1) \geqslant -2 + \frac{1}{2} + \frac{3}{5} + \frac{1}{2} \times 2 + \frac{3}{5} = \frac{1}{10}$. So x_1 is not poor, meaning that $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{3}{5} + \frac{1}{2} \times 2 + \frac{1}{10} = \frac{1}{30}$.

  * $d(u_1) \geqslant 7$. By (R2.1a), $\tau(u_1 \to u_2) \geqslant \frac{3}{5}$. Then $\omega^*(u_2) \geqslant -2 + \frac{3}{5} + \frac{3}{5} + \frac{1}{2} \times 2 = \frac{1}{5}$, and thus $\tau(u_2 \to v) \geqslant \frac{1}{10}$ by (R6.2). Hence, $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{3}{5} + \frac{1}{2} \times 2 + \frac{1}{10} = \frac{1}{30}$.

- $n_{4^+}(v_1) = 1$. Then $n_{4^+}(v_2) \geqslant 2$ by Lemma 6.17 (1). Moreover, $d(u_1) \geqslant 7$ by Lemma 6.6. By (R1.1) and (R2.1a), $\tau(v_1 \to v) \geqslant \frac{1}{4}$ and $\tau(v_2 \to v) \geqslant \frac{2}{3}$. So $\omega^*(u_2) \geqslant -2 + \frac{1}{2} + \frac{2}{3} + \frac{1}{2} \times 2 = \frac{1}{6}$. By (R6), v gets at least $\frac{1}{12}$ from u_2. We conclude that $\omega^*(v) \geqslant -2 + \frac{1}{4} + \frac{2}{3} + \frac{1}{2} \times 2 + \frac{1}{12} = 0$.

Case 3b(3): $d(v_2) = 9$. Let $N_G(v_2) = \{u_2, u_3, v, x_1, \dots, x_6\}$.

- $n_{4^+}(v_1) \geqslant 2$. Then $\tau(v_1 \to v) \geqslant \frac{1}{3}$ by (R1.1). If $n_{4^+}(v_2) \geqslant 2$, then $\tau(v_2 \to v) \geqslant \frac{5}{7}$ by (R2.1b). So $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{5}{7} + \frac{1}{2} \times 2 = \frac{1}{21}$. In what follows, assume that $n_{4^+}(v_2) = 1$. Then each x_i is a 3^--vertex for $i \in \{1, \dots, 6\}$. Moreover, by (R2.1b), $\tau(v_2 \to v) \geqslant \frac{5}{8}$.
  - If $f_{u_2v_2}$ is not bad, then $\omega^*(u_2) \geqslant -2 + \frac{1}{4} + \frac{5}{8} + \frac{1}{2} + \frac{2}{3} = \frac{1}{24}$. Thus, $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{5}{8} + \frac{1}{2} \times 2 + \frac{1}{24} = 0$ by (R6).
  - Suppose that $f_{u_2v_2}$ is bad. Let $f_{u_2v_2} = [u_1u_2v_2x_1y]$ such that $d(x_1) = 2$. By Lemma 6.1, $d(y) \geqslant 5$. It implies that $n_{4^+}(u_1) \geqslant 2$.
    * $n_{4^+}(u_1) \geqslant 3$. Then $\omega^*(u_2) \geqslant -2 + \frac{1}{2} + \frac{5}{8} + \frac{1}{2} \times 2 = \frac{1}{8}$. So $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{5}{8} + \frac{1}{2} \times 2 + \frac{1}{16} = \frac{1}{48}$ by (R6).
    * $n_{4^+}(u_1) = 2$. If $d(u_1) \geqslant 6$, then $\tau(u_1 \to u_2) \geqslant \frac{1}{2}$ by (R2.1) and (R3), and thus $\omega^*(u_2) \geqslant -2 + \frac{1}{2} + \frac{5}{8} + \frac{1}{2} \times 2 = \frac{1}{8}$. By (R6), it is easy to calculate that $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{5}{8} + \frac{1}{2} \times 2 + \frac{1}{16} = \frac{1}{48}$. Now suppose $d(u_1) = 5$. By Lemma 6.6, $d(y) \geqslant 7$. Denote by f' the other face adjacent to $f_{u_2v_2}$ distinct from f. If f' is not bad, then $\omega^*(x_1) \geqslant -2 + \frac{1}{2} + \frac{5}{8} + \frac{1}{2} + \frac{2}{3} = \frac{7}{24}$. Otherwise, assume f' is bad. Let $f' = [yx_1v_2x_2z]$ such that $d(x_2) = 2$, implying that $d(z) \geqslant 5$. Then $\omega^*(x_1) \geqslant -2 + \frac{3}{5} + \frac{5}{8} + \frac{1}{2} \times 2 = \frac{9}{40}$. So in each case, $\tau(x_1 \to u_2) \geqslant \frac{9}{80}$ by (R6). Hence, $\omega^*(u_2) \geqslant \frac{1}{3} + \frac{5}{8} + \frac{1}{2} \times 2 + \frac{9}{80} = \frac{17}{240}$. Therefore, $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{5}{8} + \frac{1}{2} \times 2 + \frac{17}{240} = \frac{7}{240}$.

- $n_{4^+}(v_1) = 1$. Then $d(u_1) \geqslant 7$ by Lemma 6.6. By (R1.1), $\tau(v_1 \to v) \geqslant \frac{1}{4}$.
  - Suppose $n_{4^+}(v_2) = 2$. Then $\tau(v_2 \to v) \geqslant \frac{5}{7}$ by (R2.1b). Since $\omega^*(u_2) \geqslant -2 + \frac{1}{2} + \frac{5}{7} + \frac{1}{2} \times 2 = \frac{3}{14}$, by (R6), we have that $\tau(u_2 \to v) \geqslant \frac{3}{28}$, and thus $\omega^*(v) \geqslant -2 + \frac{1}{4} + \frac{5}{7} + \frac{1}{2} \times 2 + \frac{3}{14} = \frac{5}{28}$.
  - Suppose $n_{4^+}(v_2) = 1$. By Lemma 6.18, $n_3(v_2) \geqslant 2$, and so $\tau(v_2 \to v) \geqslant \frac{5 - \frac{1}{2} \times 2}{9-3} = \frac{2}{3}$ by (R2.1b). Note that $\omega^*(u_2) \geqslant -2 + \frac{1}{2} + \frac{2}{3} + \frac{1}{2} \times 2 = \frac{1}{6}$. So $\tau(u_2 \to v) \geqslant \frac{1}{12}$ by (R6), and hence $\omega^*(v) \geqslant -2 + \frac{1}{4} + \frac{2}{3} + \frac{1}{2} \times 2 + \frac{1}{12} = 0$.

Case 3b(4): $d(v_2) \geqslant 10$. If $n_{4^+}(v_1) \geqslant 2$, then $\tau(v_1 \to v) \geqslant \frac{1}{3}$ by (R1.1), and thus $\omega^*(v) \geqslant -2 + \frac{1}{3} + \frac{2}{3} + \frac{1}{2} \times 2 = 0$. Otherwise, assume $n_{4^+}(v_1) = 1$. By

Lemma 6.6, we affirm that u_1 is a 7^+-vertex. At this moment, $\omega^*(u_2) \geqslant -2 + \frac{1}{2} + \frac{2}{3} + \frac{1}{2} \times 2 = \frac{1}{6}$. So $\tau(u_2 \to v) \geqslant \frac{1}{12}$ by (R6), and, therefore, $\omega^*(v) \geqslant -2 + \frac{1}{4} + \frac{2}{3} + \frac{1}{2} \times 2 + \frac{1}{12} = 0$.

This completes the proof of Theorem 6.1. $\square$

Chapter 7

An (F_1, F_4)-partition of graphs with low genus and girth at least 6

This chapter is based on the following paper:

[10] M. Chen, A. Raspaud, and W. Yu. An (F_1, F_4)-partition of graphs with low genus and girth at least 6. J. Graph Theory, 99(2):186–206, 2022.

In this chapter, we study vertex partitions of graphs under restriction on maximum average degree. Recall that the maximum average degree of G is defined to be $\text{mad}(G) = \max\{\frac{2|E(H)|}{|V(H)|} : H \subseteq G\}$. By considering sparse graphs, Borodin and Kostochka [5] obtained that every graph G satisfying $\text{mad}(G) \leqslant \frac{16}{5}$ admits a (Δ_1, Δ_4)-partition. It follows immediately that every graph in $\mathcal{PG}_6$ admits a (Δ_1, Δ_4)-partition. In this chapter, we use potential technique and discharging method to improve this result to forests partition.

For a given graph G and vertex subset $S \subseteq V(G)$, we define

$$\rho(S, G) := 8|S| - 5|E(G[S])|. \tag{7.1}$$

The main result in this chapter is the following.

Theorem 7.1. *If a graph G satisfies that*

$$\rho(S, G) > -1 \quad for \quad each \quad S \subseteq V(G), \tag{7.2}$$

then G admits an (F_1, F_4)-partition.

By definition, $\text{mad}(G) \leqslant \frac{16}{5}$ if and only if $\rho(S, G) \geqslant 0$ for each $S \subseteq V(G)$. So we deduce the following result from Theorem 7.1.

Theorem 7.2. *Every graph G with $mad(G) \leqslant \frac{16}{5}$ admits an (F_1, F_4)-partition.*

The *genus* of a graph is the minimal integer r such that the graph can be drawn without crossing edges on a sphere with r handles. Using the general Euler's formula, one can prove that for any graph G with genus r and girth at least g, the average degree, denoted $ad(G)$, satisfies:

$$ad(G) \leqslant \frac{2g}{g-2} + \frac{4g(r-1)}{(g-2)|V(G)|}.$$

This contributes to obtain the same result for planar graphs (graphs with genus 0) and toroidal graphs (graphs with genus 1) of girth at least 6. In particular, one may easily derive Corollary 7.1 from Theorem 7.2, which is a strengthening of a result in [5].

Corollary 7.1. *Every graph of genus at most* 1 *and girth at least* 6 *admits an* (F_1, F_4)-*partition.*

In Section 7.1, we give some basic notations. In Section 7.2, we use the potential technique to find the forbidden configurations in a minimum counterexample, then apply the discharging technique to obtain a contradiction in Section 7.3. Finally in Section 7.4, we give some concluding remarks.

7.1 Preliminaries

A *flag* of G is a pendant block formed by four vertices of G in which the non-cut vertices induce a $K_{1,2}$ and the cut vertex (we will call it the *base vertex* or a *host*) is adjacent to all other vertices, see Figure 7.1 (left). Meanwhile, the non-cut vertices in a flag are said to be *special vertices*. By an i-*host* we mean a vertex $v \in V(G)$ which is the base vertex of precisely i flags. A *pendant host* is defined to be a 5-host which is adjacent to exactly one non-special vertex, as shown in Figure 7.1.

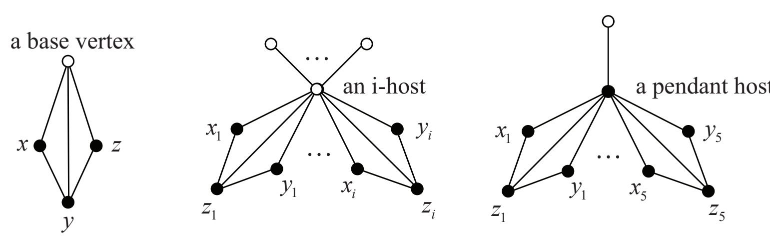

Figure 7.1: A base vertex, an i-host and a pendant host.

Obviously, every pendant host has degree 16. A *good vertex* is a 5^+-vertex which is not a pendant host. Let $i \in \{1, 4\}$, F_i is one of the two parts of the (F_1, F_4)-partition. Sometimes, we call v an F_i-*vertex* if v belongs to F_i. An F_i-*neighbour* of v is an F_i-vertex adjacent to v. Furthermore, we call v F_i-*saturated* if v is an F_i-vertex and it has exactly i F_i-neighbours. We shall denote by G^* the graph obtained from G by deleting all special vertices.

Definition 7.1. A graph H is *smaller than* a graph G if
(i) $|V_{2^+}(H^*)| < |V_{2^+}(G^*)|$, or if
(ii) $|V_{2^+}(H^*)| = |V_{2^+}(G^*)|$ and $|V(H^*)| < |V(G^*)|$, or if
(iii) $|V_{2^+}(H^*)| = |V_{2^+}(G^*)|$, $|V(H^*)| = |V(G^*)|$ and $|E(H^*)| < |E(G^*)|$, or if
(iv) $|V_{2^+}(H^*)| = |V_{2^+}(G^*)|$, $|V(H^*)| = |V(G^*)|$, $|E(H^*)| = |E(G^*)|$, and $|V(H)| < |V(G)|$.

Let G be a smallest counterexample to Theorem 7.1 in the sense of Definition 7.1. If G is disconnected, then the union of (F_1, F_4)-partitions of components of G is just an (F_1, F_4)-partition of G. Moreover, if G contains a 1-vertex, say v, then let u be the unique neighbour of v. Take an (F_1, F_4)-partition of $G - v$. We may obtain an (F_1, F_4)-partition of G by adding v to F_i such that $u \notin F_i$. Thus in what follows, we may assume that G is a connected graph with minimum degree at least 2.

7.2 Structural analysis of a minimum counterexample

Observation 7.1. *In an (F_1, F_4)-partition of a flag, there are at least two F_4-vertices.*

Claim 7.1. *Let v be a host that is incident to at least five flags. If the subgraph induced by v and all its incident flags has an (F_1, F_4)-partition, then v belongs to F_1.*

Proof. Suppose to the contrary that v is in F_4. By Observation 7.1, at least one special vertex of each flag incident to v belongs to F_4. Hence, v is adjacent to at least five F_4-vertices, a contradiction. □

Claim 7.2. *Suppose $S \subset V(G)$. The following statements hold.*

(1) *If we add a vertex $v \in V(G) - S$ to S such that v is adjacent to at least two vertices of S, then $\rho(S \cup \{v\}, G) \leqslant \rho(S, G) - 2$;*

(2) *If we add a flag H to S such that $|V(H) \cap V(S)| = 1$, then $\rho(S \cup V(H), G) = \rho(S, G) - 1$;*

Proof. (1) By definition,

$$\begin{aligned}\rho(S \cup \{v\}, G) &= 8|S \cup \{v\}| - 5|E(G[S \cup \{v\}])| \\ &\leqslant 8|S| + 8 - 5(|E(G[S])| + 2) \\ &= \rho(S, G) - 2.\end{aligned}$$

(2) By definition,

$$\rho(S \cup V(H), G) = 8|S| + 3 \times 8 - 5(|E(G[S])| + 5) = \rho(S, G) - 1.$$

This completes the proof of Claim 7.2. □

Claim 7.3. *G^* is not a complete graph K_i with $1 \leqslant i \leqslant 2$.*

Proof. Suppose to the contrary that G^* contains some K_i for $i \in \{1,2\}$. If $i = 1$, we let $K_1 = v$. Since G has no 1^--vertex, v is incident to at least one flag in G, denoted by $T_1, T_2, \ldots, T_m$. Then $G = \cup_{i=1}^{i=m} T_i$ due to the fact that G is connected. It is easy to establish an (F_1, F_4)-partition for G by adding v to F_1 and all special vertices to F_4, a contradiction. Now suppose that $i = 2$. Let uv be an edge in G^* such that u and v are incident to m and n flags in G, respectively. Similarly, $m \geqslant 1$ and $n \geqslant 1$. We can also produce an (F_1, F_4)-partition by adding u and v to F_1 and all special vertices to F_4, a contradiction. □

Claim 7.4. *For $i \geqslant 6$, G does not contain any i-host.*

Proof. Suppose to the contrary that v is an i-host for some integer $i \geqslant 6$. Let H be obtained from G by deleting the vertices (apart from v) of one incident flag based on v. By Definition 7.1, H is smaller than G. Take an (F_1, F_4)-partition of H. Observe that now v is still incident to at least five flags in H and thus $v \in F_1$ by Claim 7.1. So we may obtain an (F_1, F_4)-partition of G by adding all deleted special vertices to F_4, a contradiction. □

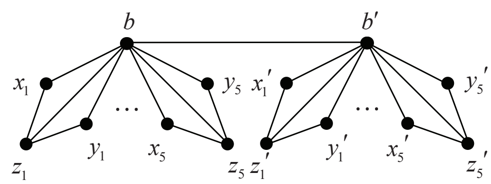

Figure 7.2: The auxiliary graph $\mathcal{B}$.

Next we have to introduce an auxiliary graph $\mathcal{B}$ where the base vertex b is a pendant host which is adjacent to the other pendant host b', as shown in Figure 7.2. First, we shall present Claim 7.5, which is quite useful and very important in the rest of paper.

Claim 7.5. *Let $S \subset V(G)$ and $\varnothing \neq S \neq V(G)$. Then $\rho(S, G) > 0$.*

Proof. Suppose to the contrary that there exists a non-empty proper subset S such that $\rho(S, G) \leqslant 0$. If $G[S]$ is disconnected, then one of its connected component, say S', satisfies that $\rho(S', G) \leqslant 0$, and thus we may choose S' instead of S. For convenience, in the following discussion, we let $H = G[S]$. The following facts 7.1-7.2 are helpful.

Fact 7.1. *The minimum degree of H is at least 2.*

Proof. Since H is connected, we may assume to the contrary that H contains a 1-vertex v. Then, $\rho(S - \{v\}, G) = 8|S - \{v\}| - 5|E(G[S - \{v\}])| = \rho(S, G) - 3 < -1$. This contradicts the assumption that $\rho(S - \{v\}, G) > -1$. □

Fact 7.2. *Each flag of G is either completely in H or disjoint from H.*

Proof. Let T be a flag of G. If $|V(T) \cap S| = 1$, then by Claim 7.2 (2), we have that $\rho(S \cup V(T), G) = \rho(S, G) - 1 \leqslant -1$, a contradiction. If $2 \leqslant |V(T) \cap S| \leqslant 3$, then there exists a vertex $v \in V(T) - S$ such that v is adjacent to at least two vertices of H. By

Claim 7.2 (1), $\rho(S \cup \{v\}, G) \leqslant \rho(S, G) - 2 < -1$, a contradiction. Hence, $|V(T) \cap S| = 0$ or $|V(T) \cap S| = 4$, verifying Fact 7.2. □

By Fact 7.2, we confirm that H is smaller than G and it satisfies the condition of Theorem 7.1 since $S \subset V(G)$. By the minimality of G, H has an (F_1, F_4)-partition. We are going to construct a graph $\widetilde{G}$ by applying following steps.

Step 1: deleting S;

Step 2: adding a copy of $\mathcal{B}$ to G;

Step 3: joining v to $b \in V(\mathcal{B})$ if $v \in V(G) - S$ and v is adjacent to an F_1-vertex $u \in S$;

Step 4: adding five distinct flags based on v if $v \in V(G) - S$ and v is adjacent to an F_4-vertex $u \in S$.

Fact 7.3. *$\widetilde{G}$ is smaller than G.*

Proof. By Fact 7.2, we see that all vertices in $G[V(G) - S]^*$ also belong to $V(\widetilde{G}^*)$. It means that each 2^+-vertex in $G[V(G) - S]^*$ must be a 2^+-vertex of $V(\widetilde{G}^*)$. Notice that $\mathcal{B}$ has exactly two vertices in $\widetilde{G}^*$. That is, b and b'. Moreover, $d_{\widetilde{G}^*}(b') = 1$ and $d_{\widetilde{G}^*}(b) \geqslant 1$. If there are at least two vertices belonging to S such that they are of degree at least 2 in G^*, then it is obvious that $|V_{2^+}(\widetilde{G}^*)| < |V_{2^+}(G^*)|$ and thus we are done. Otherwise, we have two cases to discuss:

- Assume that there is no vertex of S which is a 2^+-vertex of G^*. Then H^* is the complete graph K_i for some $i \in \{1, 2\}$. First consider the case that $H^* = K_1$. Denote by u the vertex of H^*. By Fact 7.1, we are sure that u must be incident to $t \geqslant 1$ flags in H. It follows from Claim 7.4 that $t \leqslant 5$. Thus, $\rho(S, G) = 8|S| - 5|E(G[S])| = 8 \times (3t + 1) - 5 \times 5t = 8 - t \geqslant 3$, which violates the assumption that $\rho(S, G) \leqslant 0$. Next consider the case that $H^* = K_2$. Let $H^* = xy$ with $d_{H^*}(x) = d_{H^*}(y) = 1$. Then neither x nor y can be adjacent to any vertex in $V(G) - S$, and thus xy is also a K_2 in G^*, contradicting Claim 7.3.

- Assume that there is exactly one vertex, say $u \in S$, such that $d_{G^*}(u) \geqslant 2$. In this case, we further deduce that $d_{\widetilde{G}^*}(b) \geqslant 2$; otherwise, $|V_{2^+}(\widetilde{G}^*)| = |V_{2^+}(G^*)| - 1$ and thus we are done. Now let $u_1, u_2, \ldots, u_{d_G(u)}$ denote all neighbours of u in G^*. Clearly, at most one of them belongs to H; otherwise, we deduce that $|V_{2^+}(\widetilde{G}^*)| = |V_{2^+}(G^*)|$ and $|V(\widetilde{G}^*)| < |V(G^*)|$, and thus $\widetilde{G}$ is smaller than G by Definition 7.1. Again, we deduce that $H^* = K_2$ and let $H^* = xy$. Similarly, by the above discussion, we may obtain a contradiction. □

Fact 7.4. *For any subset $A \subseteq V(\widetilde{G})$, we have that $\rho(A, \widetilde{G}) > -1$.*

Proof. In what follows, let A be the subset with minimum $\rho(A, \widetilde{G})$ satisfying that $\rho(A, \widetilde{G}) \leqslant -1$. Let $A' = A \cap V(G - H)$, $A'' = A \cap V(\mathcal{B})$ and $A''' = A - V(\mathcal{B})$. Obviously, $A' \subseteq A'''$. By the minimality of $\rho(A, \widetilde{G})$, if a flag T of G satisfies $|V(T) \cap A| \geqslant 1$, then by Claim 7.2 (1)-(2), we deduce that $V(T) \subseteq A$.

Next, let x be the number of edges that connect A' and those F_4-vertices in H, and y be the number of edges that connect A' and those F_1-vertices in H, respectively. We first derive the following:

$$\begin{aligned}\rho(A', G) &= 8(|A'''| - 15x) - 5(|E(\widetilde{G}[A'''])| - 25x) \\ &= \rho(A''', \widetilde{G}) + 5x.\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}\rho(A, \widetilde{G}) &= \rho(A'' \cup A''', \widetilde{G}) \\ &= 8|A'' \cup A'''| - 5|E(\widetilde{G}[A'' \cup A'''])| \\ &= 8|A''| + 8|A'''| - 5(|E(\widetilde{G}[A''])| + |E(\widetilde{G}[A'''])| + y) \\ &= \rho(A'', \widetilde{G}) + \rho(A''', \widetilde{G}) - 5y \\ &= \rho(A'', \mathcal{B}) + \rho(A''', \widetilde{G}) - 5y.\end{aligned}$$

Therefore,

$$\rho(A, \widetilde{G}) = \rho(A'', \mathcal{B}) + \rho(A', G) - 5x - 5y.$$

Let $W = S \cup A'$. Then $W \subseteq V(G)$. Thus,

$$\begin{aligned}\rho(W, G) &= \rho(S \cup A', G) \\ &= 8(|S| + |A'|) - 5|E(G[S \cup A'])| \\ &= 8|S| + 8|A'| - 5(|E(G[S])| + |E(G[A']) + x + y) \\ &= \rho(S, G) + \rho(A', G) - 5x - 5y \\ &= \rho(S, G) + \rho(A, \widetilde{G}) - \rho(A'', \mathcal{B}).\end{aligned}$$

At this point, one may calculate that $\rho(A'', \mathcal{B}) \geqslant 0$. Therefore, we have that $\rho(W, G) \leqslant \rho(A, \widetilde{G}) \leqslant -1$, which violates the assumption of Theorem 7.1. □

Up to now, we ensure that $\widetilde{G}$ has an (F_1, F_4)-partition. By Claim 7.1, $b \in F_1$, $b' \in F_1$, and each vertex of $V(G) - S$ which is adjacent to an F_4-vertex of S also belongs to F_1. So b is F_1-saturated, implying that each vertex of $V(G) - S$ that is both adjacent to b and an F_1-vertex of S must be in F_4. Therefore, the combination of partitions of H and $\widetilde{G}$ produces an (F_1, F_4)-partition of G, a contradiction. □

In order to avoid redundancy in proofs of Claims 7.6-7.8, we would like to give the following observation.

Observation 7.2. *Suppose that $d_{G*}(v) \geqslant 2$ such that v_1, v_2 are two of the neighbours of v in G^*. Let H be the graph obtained from $G - v$ by adding two flags T_1 and T_2 based on v_1 and v_2, respectively. Then H admits an (F_1, F_4)-partition.*

Proof. Clearly, $|V_{2^+}(H^*)| < |V_{2^+}(G^*)|$ due to the fact that v is still a 2^+-vertex in G^*. So by Definition 7.1, we have that H is smaller than G. If H does not admit any (F_1, F_4)-partition, it follows immediately that there exists a subset $S \subseteq V(H)$ such that $\rho(S, H)$ is minimum and $\rho(S, H) \leqslant -1$. Obviously, at least one special vertex of

T_1 and T_2 belongs to S since otherwise $S \subseteq V(G)$ and thus $\rho(S,H) = \rho(S,G) > -1$ by the assumption of Theorem 7.1, a contradiction. Next, let $S' = S \cap V(G)$. If $|S' \cap \{v_1, v_2\}| = 0$, then we can obtain a new vertex set, say S^*, by adding flags T_1 or T_2 to S. It follows from Claim 7.2 (1)-(2) that $\rho(S^*, H) < \rho(S,H)$, which contradicts the selection of S. If $|S' \cap \{v_1, v_2\}| = 1$, w.l.o.g., $S' \cap \{v_1, v_2\} = \{v_1\}$, then similarly by Claim 7.2 (1)-(2) and the choice of $\rho(S,H)$, we know that T_1 must be completely in S. Thus, we obtain that

$$\begin{aligned}
\rho(S', G) &= 8|S'| - 5|E(G[S'])| \\
&= 8(|S| - 3) - 5(|E(H[S])| - 5) \\
&= \rho(S,H) - 3 \times 8 + 5 \times 5 \\
&= \rho(S,H) + 1 \\
&\leqslant 0.
\end{aligned}$$

This contradicts Claim 7.5.

Next we consider the last case that $|S' \cap \{v_1, v_2\}| = 2$. Namely, v_1 and v_2 are both in S'. Then similarly we deduce that T_1 and T_2 are completely in S. Let $S^* = S' \cup \{v\}$. We have that $S^* \subset V(G)$. Then

$$\begin{aligned}
\rho(S^*, G) &= 8|S^*| - 5|E(G[S^*])| \\
&= 8(|S| - 6 + 1) - 5(|E(H[S])| - 10 + 2) \\
&= \rho(S,H) - 5 \times 8 + 40 \\
&\leqslant -1.
\end{aligned}$$

This contradicts the assumption of Theorem 7.1. Therefore, we complete the proof of Observation 7.2. □

Let $v \in V(G)$. Denote by $f(v)$ the number of flags based on v in G. For brevity, we use $d^*(v)$ instead of $d_{G^*}(v)$.

Claim 7.6. *For $v \in V(G^*)$, we have that $d^*(v) + f(v) \geqslant 3$. In particular, each 2-vertex in G is special. Besides, if $f(v) \geqslant 1$, then $d^*(v) + f(v) \geqslant 6$.*

Proof. First suppose to the contrary that G^* contains a vertex v satisfying that $d^*(v) + f(v) \leqslant 2$. By Claim 7.3, we confirm that $d^*(v) \geqslant 1$. The following discussion splits into two cases in light of the value of $d^*(v)$.

Case 1: $d^*(v) = 1$.

Then $f(v) \leqslant 1$. As $d_G(v) \geqslant 2$, we deduce that $f(v) = 1$. That is, v is a 4-vertex in G which is incident to exactly one flag, say T. Let v' be the other neighbour of v not on T. Let $H = G - V(T)$. Clearly, H is smaller than G, and H satisfies the condition of Theorem 7.1. Thus H has an (F_1, F_4)-partition. If $v' \in F_1$, then we add v and both special 2-vertices of T to F_4 and the special 3-vertex of T to F_1. Otherwise, assume that $v' \in F_4$. We add v to F_1 and all remaining special vertices of T to F_4. In each case, one may verify that the obtained partition of G is an (F_1, F_4)-partition, a contradiction.

Case 2: $d^*(v) = 2$.

Then $f(v) = 0$, implying that $d_G(v) = 2$. Let $N_G(v) = \{v_1, v_2\}$. Let H be the graph obtained from $G - v$ by adding two flags T_1 and T_2 based on v_1 and v_2, respectively. By Observation 7.2, H has an (F_1, F_4)-partition. Notice that for each $i \in \{1, 2\}$ there is at least one vertex of T_i (apart from v_i) belonging to F_4. So in order to obtain an (F_1, F_4)-partition of G, it suffices to add v to F_4 if at least one of v_1 and v_2 belongs to F_1, and add v to F_1 otherwise.

Hence, we conclude that $d^*(v) + f(v) \geqslant 3$, which guarantees us that each 2-vertex in G must be special.

Next suppose that $f(v) \geqslant 1$. If $d^*(v)+f(v) \leqslant 5$, let $T_1, \ldots, T_{f(v)}$ be the incident flags based on v and denote by $v_1, \ldots, v_{d*(v)}$ the other neighbours of v which are not lying on T_i. Let $H = G - (\cup_{i=1}^{i=f(v)} V(T_i) - v)$. Apparently, H is smaller than G, and it satisfies the condition of Theorem 7.1. So by the minimality of G, H has an (F_1, F_4)-partition. If $v \in F_1$, then we can add all the deleted vertices of $\cup_{i=1}^{i=f(v)} V(T_i)$ to F_4. Otherwise, assume that $v \in F_4$. If $v_1, \ldots, v_{d*(v)}$ are all in F_4, then we can first change v to F_1 and then go back to the previous case. Otherwise, there exists some $v_i \in F_1$. In this case, we add all deleted special 3-vertices to F_4 and other special 2-vertices to F_1. Since $d^*(v) + f(v) \leqslant 5$ and some $v_i \in F_1$, one may easily check that the obtained partition of G is an (F_1, F_4)-partition, a contradiction. □

Corollary 7.2. *Each special vertex has a base vertex of degree at least* 8.

Proof. Let v be a special vertex of G whose base vertex is u. Then $f(u) \geqslant 1$. By Claim 7.6, we deduce that $d_G(u) = d^*(u) + 3f(u) \geqslant 6 + 2f(u) \geqslant 8$. □

For convenience, let $ph(v)$ denote the number of pendant hosts adjacent to a vertex $v \in V(G)$.

Claim 7.7. *If v is a non-special* 3*-vertex, then $ph(v) = 0$.*

Proof. Suppose that $d_G(v) = 3$ and $N_G(v) = \{v_1, v_2, v_3\}$, where v_1 is a pendant host such that v_1 is incident to exactly five flags $T_1, \ldots, T_5$. Let H be obtained from $G - v$ by adding two flags T_2^* and T_3^* based on v_2 and v_3, respectively. By Observation 7.2, H admits an (F_1, F_4)-partition. Moreover, by Claim 7.1, we know that $v_1 \in F_1$. We are going to show an (F_1, F_4)-partition of G, which contradicts our assumption. If at most one of v_2 and v_3 belongs to F_4, then we add v to F_4. Otherwise, assume that $v_2 \in F_4$ and $v_3 \in F_4$. At this point, we only need to first add v to F_1, and then move all the special vertices of each T_i to F_4. □

For a non-special 3-vertex u, if u has i (resp. at least i) 4^+-neighbours in G, then u is said to be a 3^i*-vertex* (resp. 3^{i^+}*-vertex*). Moreover, if u is adjacent to one 3^0-vertex (resp. two 3^0-vertices), then we call u a *weak 3-vertex* (resp. *bad* 3*-vertex*). In what follows, a vertex w is called *heavy* if it is adjacent to at least one 6^+-vertex which is incident to at most four flags.

Claim 7.8. *Suppose that v is a 3^0-vertex in G. Then at least two of its neighbours are heavy.*

Proof. Let v be adjacent to v_1, v_2, v_3 such that each v_i is a 3^--vertex in G. Notice that $d_G(v_i) \neq 1$ for all $i = 1, 2, 3$. If there is some v_i having degree 2, say $d_G(v_1) = 2$, then by Claim 7.6, v_1 is a special 2-vertex, and thus v_2 and v_3 should be lying on the flag. This contradicts Corollary 7.2.

Next, assume that all neighbours of v are 3-vertices. Let $N(v_i) = \{v, u_{2i-1}, u_{2i}\}$ for each $i \in \{1, 2, 3\}$. Suppose to the contrary that there is at most one heavy vertex among v_1, v_2 and v_3. In other words, at least two neighbours of v, say v_2 and v_3, are not heavy. So only v_1 might be heavy. Let H be obtained from $G - v_1$ by adding two flags T_1 and T_2 based on u_1, u_2, respectively. Since $d^*(v_1) \geqslant 3$, by Observation 7.2, we know that H admits an (F_1, F_4)-partition.

Next, we are going to establish an (F_1, F_4)-partition for G. Let $S = \{v, u_1, u_2\}$. If at most one vertex of S belongs to F_4, then it suffices to add v_1 to F_4. If all vertices of S belong to F_4, then we may add v_1 to F_1. Otherwise, assume that exactly two vertices of S are in F_4. So only one vertex of S belongs to F_1. All that remains is to discuss two cases below.

- $v \in F_1$ and $u_1, u_2 \in F_4$. Then, at most one of v_2 and v_3 belongs to F_1. If v_2 and v_3 are both in F_4, then one may directly add v_1 to F_1. Or else, assume w.l.o.g., that $v_2 \in F_1$ and $v_3 \in F_4$. Then we first change v to F_4 and then add v_1 to F_1.

- $u_1 \in F_1$ and $v, u_2 \in F_4$. If $v_2, v_3 \in F_1$, then add v_1 to F_4. If $v_2, v_3 \in F_4$, then first change v to F_1 and then add v_1 to F_4. Now assume that $v_2 \in F_1$ and $v_3 \in F_4$. Notice that at most one of u_3 and u_4 belongs to F_1. If u_3 and u_4 are both in F_4, then we may change v to F_1 and then add v_1 to F_4. Otherwise, w.l.o.g., assume that $u_3 \in F_4$ and $u_4 \in F_1$. Since v_2 is not heavy, by Claim 7.1, u_3 must be a 5^--vertex. Then we can change v_2 to F_4, v to F_1, and add v_1 to F_4. If the obtained partition of G is not as required, then the unique possible case is that $d_G(u_3) = 5$ and u_3 becomes F_4-saturated. Therefore, we may further change u_3 to F_1.

In each of the above cases, one can easily verify that the obtained partition of G is an (F_1, F_4)-partition. □

Claim 7.9. *If $d_G(v) = 4$, then $ph(v) \leqslant 2$.*

Proof. Suppose to the contrary that G has a 4-vertex v adjacent to at least three pendant hosts v_1, v_2 and v_3. Let v_4 be the neighbour of v different from v_1, v_2 and v_3. Let H be the graph obtained from $G - v$ by adding one flag T based on v_4. Clearly, by Definition 7.1, H is smaller than G.

First, we show that H satisfies the condition of Theorem 7.1. Suppose that there exists a subset $S \subseteq V(H)$ such that $\rho(S, H)$ is minimum and $\rho(S, H) \leqslant -1$. Then at least one special vertex of T belongs to S. In the following, let $S' = S \cap V(G)$. By Claim 7.2 (1)-(2) and the choice of $\rho(S, H)$, we know that all the vertices of T are completely in S. Thus, we obtain that

$$\begin{aligned}\rho(S', G) &= 8|S'| - 5|E(G[S'])| \\ &= 8(|S| - 3) - 5(|E(H[S])| - 5) \\ &= \rho(S, H) + 1 \\ &\leqslant 0.\end{aligned}$$

This contradicts Claim 7.5, and hence H satisfies the condition of Theorem 7.1 and then it admits an (F_1, F_4)-partition. By Claim 7.1, v_1, v_2 and v_3 all belong to F_1. So one can easily add v to F_4 to construct an (F_1, F_4)-partition of G, a contradiction. □

The following Claim 7.10-Claim 7.13 will play an important role in discharging argument in Section 3.3. For simplicity, we use $n_k(v)$ to denote the number of k-neighbours of v.

Claim 7.10. *Let $v \in V(G)$. Suppose that $d^*(v) = 2$ and $f(v) = 4$. Then $ph(v) = n_3(v) = 0$.*

Proof. Let v_1 and v_2 be two neighbours of v in G^*. Suppose, w.l.o.g., to the contrary that v_1 is either a pendant host or a 3-vertex. The discussion is divided into two cases below:

Case 1: v_1 is a pendant host.

By definition, v_1 is incident to exactly five flags. Let H be the graph obtained from G by deleting v_1 and all its incident flags. Then H is a subgraph of G and thus H satisfies the condition of Theorem 7.1. So by the minimality of G, H has an (F_1, F_4)-partition. Then one may add v_1 to F_1 and all deleted special neighbours to F_4. If the resultant partition is not an (F_1, F_4)-partition of G, then we deduce that v is F_1-saturated. Namely, $v \in F_1$ and v has one F_1-neighbour. If $v_2 \in F_1$, then we change v to F_4, and all special 3-neighbours and special 2-neighbours of v to F_4 and F_1, respectively. Otherwise, assume that $v_2 \in F_4$. At this point, it suffices to change all special neighbours of v to F_4.

Case 2: v_1 is a 3-vertex.

Denote by u_1, u_2 two neighbours of v_1 distinct from v. Let H be the graph obtained from $G - v_1$ by adding one flag based on u_1, u_2 and v, respectively. For convenience, we use the notation T_1, T_2, T_3 to represent these new added flags incident to u_1, u_2 and v, respectively. By Definition 7.1, we know that H is smaller than G.

Next, we show that H satisfies the condition of Theorem 7.1. Suppose to the contrary that there exists a subset $S \subseteq V(H)$ such that $\rho(S, H)$ is minimum and $\rho(S, H) \leqslant -1$. Then for each $i \in \{1, 2, 3\}$, at least one special vertex of T_i belongs to S. In the following, let $S' = S \cap V(G)$. There are fours cases to discuss.

- $|S' \cap \{u_1, u_2, v\}| = 0$. Then we can get a subgraph with smaller ρ by Claim 7.2 (1)-(2), a contradiction.

- $|S' \cap \{u_1, u_2, v\}| = 1$. W.l.o.g., assume that $S' \cap \{u_1, u_2, v\} = \{v\}$. By the choice of $\rho(S, H)$, we know that T_3 is completely in S. Thus, we obtain that

$$\begin{aligned} \rho(S', G) &= 8|S'| - 5|E(G[S'])| \\ &= 8(|S| - 3) - 5(|E(H[S])| - 5) \\ &= \rho(S, H) - 24 + 25 \\ &= \rho(S, H) + 1 \\ &\leqslant 0. \end{aligned}$$

  This contradicts Claim 7.1.

- $|S' \cap \{u_1, u_2, v\}| = 2$. W.l.o.g., assume that $S' \cap \{u_1, u_2, v\} = \{u_1, u_2\}$. Then both T_1 and T_2 are completely in S. Let $S^* = S' \cup \{v_1\}$ and thus $S^* \subset V(G)$. Then

$$\begin{aligned} \rho(S^*, G) &= 8|S^*| - 5|E(G[S^*])| \\ &= 8(|S| - 6 + 1) - 5(|E(H[S])| - 10 + 2) \\ &= \rho(S, H) - 40 + 40 \\ &\leqslant -1. \end{aligned}$$

  This contradicts the assumption of Theorem 7.1,

- $|S' \cap \{u_1, u_2, v\}| = 3$. That is, $\{u_1, u_2, v\} \subset S'$. By the choice of $\rho(S, H)$, we know that T_1, T_2 and T_3 are completely in S. Let $S^* = S' \cup \{v_1\}$. Then

$$\begin{aligned} \rho(S^*, G) &= 8|S^*| - 5|E(G[S^*])| \\ &= 8(|S| - 9 + 1) - 5(|E(H[S])| - 15 + 3) \\ &= \rho(S, H) - 64 + 60 \\ &= \rho(S, H) - 4 \\ &< -1. \end{aligned}$$

  This contradicts the assumption of Theorem 7.1.

In each case, one may always obtain a contradiction, and hence H satisfies the condition of Theorem 7.1, meaning that H admits an (F_1, F_4)-partition. Now we show that the (F_1, F_4)-partition of H can be extended to G. Note that $v \in F_1$ by Claim 7.1, since v is incident to five flags in H. If at most one of u_1 and u_2 belongs to F_4, then we can add v_1 to F_4. Otherwise, assume that $u_1, u_2 \in F_4$. If $v_2 \in F_4$, we can add v_1 to F_1 and change all v's special neighbours to F_4. Otherwise, assume that $v_2 \in F_1$. Here, one may first move v to F_4, then change all v's special 3-neighbours to F_4 and all v's special 2-neighbours to F_1. Finally v_1 can be added to F_1 preserving its property. □

In what follows, let $n^*_{3^i}(v)$ be the number of 3^i-neighbours of v in G^*. A similar definition can be given for $n^*_{3^{i+}}(v)$.

Claim 7.11. *Let $v \in V(G)$. Suppose that $d^*(v) = 3$ and $f(v) = 3$. If $n^*_{3^1}(v) \geqslant 1$, then v is adjacent to at least one good 6^+-vertex.*

Proof. Let v_1, v_2 and v_3 be all neighbours of v in G^*. W.l.o.g., assume that v_1 is a 3^1-vertex such that two neighbours of v_1, say u_1, u_2, are 3-vertices. Recall that a good 6^+-vertex is a 6^+-vertex which is not a pendant host. So next, suppose to the contrary that for $i \in \{2, 3\}$, each v_i is either a 5^--vertex or a pendant host. Let H be the graph obtained from $G - v_1$ by adding one flag T based on v. Clearly, $|V_{2^+}(H^*)| < |V_{2^+}(G^*)|$. By Definition 7.1, H is smaller than G.

First, we prove that H satisfies the condition of Theorem 7.1. Suppose to the contrary that there is a subset $S \subseteq V(H)$ such that $\rho(S, H)$ is minimum and $\rho(S, H) \leqslant -1$. It is obvious that at least one special vertex of T belongs to S. Denote $S' = S \cap V(G)$. If $v \notin S'$, then we can get a smaller ρ by Claim 7.2 (1)-(2), violating our choice of S. Thus, $v \in S'$. Moreover, by the selection of $\rho(S, H)$, we deduce that T is completely in S. Thus, we have that

$$\begin{aligned}
\rho(S', G) &= 8|S'| - 5|E(G[S'])| \\
&= 8(|S| - 3) - 5(|E(H[S])| - 5) \\
&= \rho(S, H) - 24 + 25 \\
&= \rho(S, H) + 1 \\
&\leqslant 0.
\end{aligned}$$

This contradicts Claim 7.5. So H admits an (F_1, F_4)-partition.

Next, we are going to show that the (F_1, F_4)-partition of H can be extended to G. Let $S = \{u_1, u_2, v\}$. If all vertices of S belong to F_4, then we add v_1 to F_1. If at most one vertex of S belongs to F_4, then we add v_1 to F_4. Otherwise, assume that exactly two vertices of S are in F_4. In other words, exactly one vertex of S is in F_1.

- First suppose that $u_1 \in F_1$ and $u_2, v \in F_4$. Notice that v is incident to four flags in H. By Observation 7.1, we know that v has four F_4-neighbours in H, and thus we are sure that both v_2 and v_3 are in F_1. So we can add v_1 to F_4 successfully.

- Now suppose that $v \in F_1$ and $u_1, u_2 \in F_4$. If $v_2, v_3 \in F_4$, then we can add v_1 to F_1 and change all special neighbours of v to F_4. Otherwise, w.l.o.g., assume that $v_2 \in F_1$ and $v_3 \in F_4$. Since each pendant host belongs to F_1 by Claim 7.1, we deduce that v_3 is a 5^--vertex in G. In this case, we change v to F_4, all v's special 3-neighbours to F_4 and all v's special 2-neighbours to F_1. If the obtained partition of G is not as desired, then the unique possible case is that $d_G(v_3) = 5$ and v_3 has five F_4-neighbours. So we may continue to change v_3 to F_1. □

Claim 7.12. *Let $v \in V(G)$. Suppose that $d^*(v) = 4$ and $f(v) = 2$. Then $ph(v) = 0$.*

Proof. Suppose to the contrary that there is a neighbour u of v such that u is a pendant host in G^*. Let H be the graph obtained by deleting the incident two flags based on v

(apart from v). Then H is smaller than G and satisfies the condition of Theorem 7.1. Hence, H admits an (F_1, F_4)-partition due to the minimality of G.

If $v \in F_1$, then we add all the deleted special neighbours of v to F_4. Now suppose that $v \in F_4$. Since u is a pendant host, u belongs to F_1 by Claim 7.1. So if v has exactly three F_4-neighbours in H, then one may first change v to F_1, all special neighbours of u to F_4, and finally add all the deleted special neighbours of v to F_4. Otherwise, assume that v has at most two F_4-neighbours in H. In this case, we may add two special 3-neighbours of v to F_4 and four special 2-neighbours of v to F_1. In each case, one may inspect that the obtained partition is an (F_1, F_4)-partition of G, a contradiction. □

Claim 7.13. *Let $v \in V(G)$ with $d^*(v) + f(v) = 6$. If $f(v) \in \{1, 2\}$ and $n^*_{4^+}(v) = 0$, then the following hold:*
(1) $n^*_{3^{2+}}(v) \geqslant 1$;
(2) If $n^*_{3^{2+}}(v) = 1$, then the unique 3^{2^+}-neighbour of v is adjacent to at least two good 6^+-vertices.

Proof. Let $N_{G^*}(v) = \{v_1, \ldots, v_{d^*(v)}\}$ such that v_i is a 3-vertex for each $i \in \{1, \ldots, d^*(v)\}$. For each v_1, let u_{1i} and u_{2i} denote the two neighbours of v_i distinct from v. Let H be the graph obtained from G by deleting all flags (apart from v) based on v. Then H is smaller than G and it satisfies the condition of Theorem 7.1. Thus, H admits an (F_1, F_4)-partition due to the minimality of G.

If $v \in F_1$, then we can directly add all the deleted special neighbours to F_4. Suppose now that $v \in F_4$. If at least two vertices of $N_{G^*}(v)$ belong to F_1, then it is easy to add the deleted special 3-neighbours of v to F_4 and the deleted special 2-neighbours to F_1. If all vertices of $N_{G^*}(v)$ belong to F_4, then we first move v to F_1 and then add all the deleted special neighbours to F_4. So next, we may consider the case that exactly one vertex of $N_{G^*}(v)$ belongs to F_1. W.l.o.g., suppose that $v_1 \in F_1$ and $v_2, \ldots, v_{d^*(v)}$ are all in F_4. If v_1 is not F_1-saturated, namely, $u_{11}, u_{21} \in F_4$, then move v to F_1 and we may go back to the previous case. Otherwise, assume that v_1 is F_1-saturated. It means that exactly one of u_{11} and u_{21} belongs to F_1. Next, we will make use of contradictions to show (1) and (2).

(1) Suppose to the contrary that for all $i \in \{1, \ldots, d^*(v)\}$, v_i is a 3^1-vertex. So both u_{11} and u_{21} are 3^--vertices. W.l.o.g., assume that $u_{11} \in F_1$ and $u_{21} \in F_4$. Then we can change v_1 to F_4, v to F_1, and finally add all the deleted special neighbours to F_4 successfully.

(2) Let v_j be the unique 3^{2^+}-neighbour of v. If $j \neq 1$, then v_1 is still a 3^1-vertex and thus the argument is the same as the previous case. Now assume that $j = 1$. By definition, at least one of u_{11} and u_{21} is a 4^+-vertex, say $d_G(u_{11}) \geqslant 4$. Since v is a good 6^+-vertex, in order to show (2), we will prove that u_{11} is a good 6^+-vertex. Suppose to the contrary that u_{11} is either a 5^--vertex or a pendant host. First, we move v_1 to F_4 and v to F_1. If the resulting partition is not an (F_1, F_4)-partition, then it must be the case that $u_{11} \in F_4$ and $u_{21} \in F_1$. By Claim 7.1, we deduce that u_{11} is a 5-vertex. Furthermore, u_{11} has exactly five F_4-neighbours in G. At this point, one may change

u_{11} to F_1 and add all the deleting special neighbours to F_4. It is easy to verify that the current partition is an (F_1, F_4)-partition of G. □

Claim 7.14. *Let v be a 4-vertex in G such that $n_3(v) + ph(v) = 4$. If there is at most one 3^{2^+}-vertex adjacent to v, then G contains at least one good 5^+-vertex.*

Proof. Let $v_1, \ldots, v_4$ denote all neighbours of v. By Claim 7.9, $ph(v) \leqslant 2$. Then $n_3(v) \geqslant 2$ due to the assumption that $n_3(v) + ph(v) = 4$. W.l.o.g., assume that $d_G(v_1) = d_G(v_2) = 3$. Denote by x_i and y_i the two neighbours of v_i for $i \in \{1, 2\}$. Since there is at most one 3^{2^+}-neighbour of v, we may further assume that v_1 is a 3^1-vertex. Let $H = G - v$. Obviously, H is smaller than G and it satisfies the condition of Theorem 7.1. Hence, H admits an (F_1, F_4)-partition.

Let $S = \{v_1, v_2, v_3, v_4\}$. By Claim 7.1, each pendant host belongs to F_1. So if at most one vertex of S belongs to F_4, then this vertex must be a 3-vertex, and thus we can add v to F_4. Next suppose that there is at most one vertex of S belonging to F_1, say $v_i \in F_1$. If v_i is a 3-vertex, then one can add v to F_1. If the obtained partition is not an (F_1, F_4)-partition, then we deduce that one of v_i's neighbours, denoted by x_i, belongs to F_4 and is of degree at least 5. Otherwise we put v_i in F_4 since it has at most three F_4-neighbours if it has degree at most 4 and furthermore y_i belongs to F_1. Hence, x_i is a good 5^+-vertex by Claim 7.1. Otherwise, v_i is a pendant host. In this case, we can first change all special vertices of v_i to F_4 and then add v to F_1. Now suppose that exactly two vertices of S belong to F_1. Namely, $v_i, v_j \in F_1$ and $v_k, v_m \in F_4$, where $\{i, j, k, m\} = \{1, 2, 3, 4\}$. Again, by Claim 7.1, we confirm that v_i, v_j are pendant hosts. Then $\{i, j\} = \{3, 4\}$ and thus $v_1, v_2 \in F_4$. In order to establish an (F_1, F_4)-partition of G, we have three possibilities to handle.

- If $x_1, y_1 \in F_1$, then add v to F_4.
- If $x_1, y_1 \in F_4$, then move v_1 to F_1 and add v to F_4.
- Now assume that $x_1 \in F_1$ and $y_1 \in F_4$, which implies that x_1 and v_2 cannot be the same vertex. Since v_1 is a 3^1-vertex, we see that $d_G(x_1) \leqslant 3$. Moreover, x_1 cannot be a 2-vertex by Claim 7.6. Let x_1' and x_1'' denote the neighbours of x_1 distinct from v_1. Clearly, if x_1 is not F_1-saturated, then one may easily change v_1 to F_1 and then add v to F_4. Otherwise, assume w.l.o.g., that $x_1' \in F_1$ and $x_1'' \in F_4$. Notice that x_1'' is distinct from v, since x_1 is in F_1 and v_2 is in F_4. Then x_1'' cannot be a pendant host by Claim 7.1. In this case, we change x_1 to F_4, v_1 to F_1 and finally add v to F_4. If the resultant partition is not as desired, then x_1'' must be a good 5^+-vertex and thus we are done. □

7.3 Discharging procedure

We are now ready to present a discharging procedure that will complete the proof of Theorem 7.1. For $v \in V(G)$, an initial charge function ω is defined to be $\omega(v) = d(v) - \frac{16}{5}$.

By the relation $\sum_{v\in V(G)} d(v) = 2|E(G)|$ and the assumption of Theorem 7.1, we know that the total sum of charges of the vertices satisfies the following

$$\sum_{v\in V(G)} \omega(v) = \sum_{v\in V(G)} (d_G(v) - \frac{16}{5}) < \frac{2}{5}.$$

Let $\omega^*(v)$ be the charge of $v \in V(G)$ after the discharge procedure. In order to lead to a contradiction, we shall prove that $\sum_{v\in V(G)} \omega^*(v) \geqslant \frac{2}{5}$.

Let $\tau(u \to v)$ denote the amount of charges transferring from u to v. Below are the discharging rules:

(R1) Every special 2-vertex and special 3-vertex respectively gets $\frac{6}{5}$ and $\frac{1}{5}$ from its 8^+-neighbour.
(R2) Every 4-vertex sends $\frac{1}{5}$ and $\frac{1}{10}$ to each 3^1-neighbour and 3^{2^+}-neighbour, respectively.
(R3) Every 5-vertex sends $\frac{1}{5}$ to each of its 3-neighbour.
(R4) Let u be a 6^+-vertex in G adjacent to a 3-vertex v.
  (R4.1) Assume that v is a 3^1-vertex. Then
    (R4.1.1) $\tau(u \to v) = \frac{2}{5}$ if v is bad;
    (R4.1.2) $\tau(u \to v) = \frac{3}{10}$ if v is weak;
    (R4.1.3) $\tau(u \to v) = \frac{1}{5}$ otherwise.
  (R4.2) Assume that v is a 3^{2^+}-vertex. Then
    (R4.2.1) $\tau(u \to v) = \frac{1}{5}$ if u is weak;
    (R4.2.2) $\tau(u \to v) = \frac{1}{10}$ otherwise.
(R5) Every 3^0-vertex gets $\frac{1}{10}$ from each of its 3-neighbour which is adjacent to at least one 6^+-vertex.
(R6) Every pendant host gets $\frac{1}{5}$ from its 4^+-neighbour.

7.3.1 Find charges are nonnegative

In this subsection, we will show that $\omega^*(v) \geqslant 0$ for each $v \in V(G)$. Notice that $d_G(v) \geqslant 2$.

- $d_G(v) = 2$. Then $\omega(v) = -\frac{6}{5}$. By Claim 7.6, v is a special 2-vertex. Moreover, the base vertex of v is of degree at least 8 by Corollary 7.2. So by (R1), we have that $\omega^*(v) = -\frac{6}{5} + \frac{6}{5} = 0$.

- $d_G(v) = 3$. Then $\omega(v) = -\frac{1}{5}$. If v is special, then similarly, by Corollary 7.2 and (R1), we obtain that $\omega^*(v) = -\frac{1}{5} + \frac{1}{5} = 0$. Now assume that v is not a special vertex. Namely, it is not incident to any flags. Let v be a 3^i-vertex.

  - $i = 0$. By Claim 7.8, v has at least two heavy 3-neighbours. By definition, these two heavy 3-neighbours both have at least one 6^+-neighbour. So $\omega^*(v) \geqslant -\frac{1}{5} + 2 \times \frac{1}{10} = 0$ by (R5).
  - $i = 1$. Let v_1 be the unique 4^+-neighbour of v. If $d_G(v_1) \in \{4, 5\}$, then both (R2) and (R3) guarantee that $\tau(v_1 \to v) = \frac{1}{5}$ and thus $\omega^*(v) = -\frac{1}{5} + \frac{1}{5} = 0$.

Otherwise, assume that $d_G(v_1) \geqslant 6$. If v is bad, namely, v has two 3^0-neighbours, then by (R4.1.1) and (R5) we have that $\omega^*(v) \geqslant -\frac{1}{5} + \frac{2}{5} - 2 \times \frac{1}{10} = 0$. If v is weak, namely, v has one 3^0-neighbour, then by (R4.1.2) and (R5) we have that $\omega^*(v) \geqslant -\frac{1}{5} + \frac{3}{10} - \frac{1}{10} = 0$. Otherwise, one may easily deduce that $\omega^*(v) \geqslant -\frac{1}{5} + \frac{1}{5} = 0$ by (R4.1.3).

– $i \geqslant 2$. Let v_1, v_2, v_3 denote all neighbours of v such that $d_G(v_1), d_G(v_2) \geqslant 4$. If $d_G(v_3) \geqslant 4$, then $\omega^*(v) = -\frac{1}{5} + 3 \times \frac{1}{10} = \frac{1}{10}$ by (R2). Suppose $d_G(v_3) \leqslant 3$. If $d_G(v_1), d_G(v_2) \leqslant 5$, then similarly we deduce that $\omega^*(v) = -\frac{1}{5} + 2 \times \frac{1}{10} = 0$ by (R2) or $\omega^*(v) = -\frac{1}{5} + \frac{1}{5} = 0$ by (R3). Otherwise, by symmetry assume that $d_G(v_1) \geqslant 6$. If v_3 is a 3^0-vertex, then v is weak, and thus $\tau(v_1 \to v) = \frac{1}{5}$ by (R4.2.1) and $\tau(v \to v_3) = \frac{1}{10}$ by (R5). Moreover, $\tau(v_2 \to v) \geqslant \frac{1}{10}$ by (R2) and (R3). So $\omega^*(v) \geqslant -\frac{1}{5} + \frac{1}{5} + \frac{1}{10} - \frac{1}{10} = 0$. Or else, it is not difficult to deduce that $\omega^*(v) \geqslant -\frac{1}{5} + 2 \times \frac{1}{10} = 0$ by (R2), (R3) and (R4.2.2).

- $d_G(v) = 4$. Then $\omega(v) = \frac{4}{5}$. By (R2), v sends each neighbour at most $\frac{1}{5}$. Thus, $\omega^*(v) \geqslant \frac{4}{5} - 4 \times \frac{1}{5} = 0$.

- $d_G(v) = 5$. Then $\omega(v) = \frac{9}{5}$. By (R3), we see that $\omega^*(v) \geqslant \frac{9}{5} - 5 \times \frac{1}{5} = \frac{4}{5}$.

- $d_G(v) \geqslant 6$. Obviously, $v \in V(G^*)$. If $f(v) = 0$, then $\omega^*(v) \geqslant d_G(v) - \frac{16}{5} - \frac{2}{5}d_G(v) = \frac{3}{5}d_G(v) - \frac{16}{5} \geqslant \frac{2}{5}$ by (R4) and (R6). Otherwise, by Claim 7.4, we have that $1 \leqslant f(v) \leqslant 5$. At this point, one may obtain that $f(v) + d^*(v) \geqslant 6$ by Claim 7.6. This fact will be used later frequently without any special mention. Moreover, by (R1), we observe that v sends a charge of $2 \times \frac{6}{5} + \frac{1}{5} = \frac{13}{5}$ to all special vertices incident to the same flag. Since $d_G(v) = d^*(v) + 3f(v)$, v has a charge of at least $d^*(v) + 3f(v) - \frac{16}{5} - \frac{13}{5}f(v)$ which equals $d^*(v) - \frac{16}{5} + \frac{2}{5}f(v)$ after transferring charges to all its adjacent special vertices. For shortness, we define

$$\sigma(v) = d^*(v) - \frac{16}{5} + \frac{2}{5}f(v).$$

There are four cases that need to be handled.

– Assume $f(v) = 1$. If $d^*(v) \geqslant 6$, then by (R4), we have that $\omega^*(v) \geqslant \sigma(v) - \frac{2}{5}d^*(v) = \frac{3}{5}d^*(v) - \frac{14}{5} \geqslant \frac{4}{5}$. Otherwise, assume that $d^*(v) = 5$. Then $\sigma(v) = \frac{11}{5}$. By Claim 7.13 (1), we see that either $n^*_{4^+}(v) \geqslant 1$ or $n^*_{4^+}(v) = 0$ and $n^*_{3^{2+}}(v) \geqslant 1$. By (R4.2) and (R6), we know that at least one neighbour in G^* of v takes charge at most $\frac{1}{5}$ from v. Thus, $\omega^*(v) \geqslant \frac{11}{5} - \frac{1}{5} - 4 \times \frac{2}{5} = \frac{2}{5}$.

– Assume $f(v) = 2$. If $d^*(v) \geqslant 5$, then $\omega^*(v) \geqslant \sigma(v) - \frac{2}{5}d^*(v) = \frac{3}{5}d^*(v) - \frac{12}{5} \geqslant \frac{3}{5}$ by (R4). Otherwise, assume that $d^*(v) = 4$. Then $\sigma(v) = \frac{8}{5}$. Similarly, by Claim 7.13 (1), v is either adjacent to at least a 4^+-vertex or a 3^{2^+}-vertex in G^*. Therefore, $\omega^*(v) \geqslant \frac{8}{5} - \frac{1}{5} - 3 \times \frac{2}{5} = \frac{1}{5}$ by (R4) and (R6).

– Assume $f(v) = 3$. If $d^*(v) \geqslant 4$, then $\omega^*(v) \geqslant \sigma(v) - \frac{2}{5}d^*(v) = \frac{3}{5}d^*(v) - \frac{10}{5} \geqslant \frac{2}{5}$ by (R4). Otherwise, suppose that $d^*(v) = 3$. Then $\sigma(v) = 1$. If $n^*_{3^1}(v) = 0$, then by (R4.2) we have that $\omega^*(v) \geqslant 1 - \frac{1}{5} \times 3 = \frac{2}{5}$. Or else, suppose that

$n^*_{3^1}(v) \geqslant 1$. On the one hand, by (R4.1), such a 3^1-neighbour gets charge at most $\frac{2}{5}$ from v. On the other hand, v is adjacent to at least one good 6^+-vertex by Claim 7.11. By (R1)-(R6), this kind of good 6^+-vertex does not obtain any charges from v. Hence, $\omega^*(v) \geqslant 1 - \frac{2}{5} \times 2 = \frac{1}{5}$.

- Assume $f(v) = 4$. By (R4), $\omega^*(v) \geqslant \sigma(v) - \frac{2}{5}d^*(v) = \frac{3}{5}d^*(v) - \frac{8}{5} \geqslant \frac{4}{5}$ if $d^*(v) \geqslant 4$ and $\omega^*(v) \geqslant \sigma(v) - \frac{2}{5}d^*(v) = \frac{3}{5}d^*(v) - \frac{8}{5} \geqslant \frac{1}{5}$ if $d^*(v) = 3$. Now consider the case that $d^*(v) = 2$. By Claim 7.10, we are sure that v cannot be adjacent to any pendant hosts or 3-vertices. Therefore, $\omega^*(v) \geqslant \sigma(v) \geqslant \frac{2}{5}$.
- Assume $f(v) = 5$. If $d^*(v) \geqslant 3$, by (R4), $\omega^*(v) \geqslant \sigma(v) - \frac{2}{5}d^*(v) = \frac{3}{5}d^*(v) - \frac{6}{5} \geqslant \frac{3}{5}$. If $d^*(v) = 1$, namely, v itself is a pendant host, then $\sigma(v) = -\frac{1}{5}$. By (R6), we see that $\omega^*(v) \geqslant -\frac{1}{5} + \frac{1}{5} = 0$. Now consider the final case that $d^*(v) = 2$. Then $\sigma(v) = \frac{4}{5}$. If $n^*_{3^1}(v) \leqslant 1$, then $\omega^*(v) \geqslant \frac{4}{5} - \frac{2}{5} - \frac{1}{5} = \frac{1}{5}$ by (R4) and (R6). Otherwise, we deduce that $\omega^*(v) \geqslant \frac{4}{5} - \frac{2}{5} \times 2 = 0$ by (R4.1).

7.3.2 Total charge

Now we are going to show that $\sum_{v \in V(G)} \omega^*(v) \geqslant \frac{2}{5}$. First, we observe the following.

Observation 7.3. *G contains at least one good 4^+-vertex.*

Proof. Let $v \in V(G)$. Clearly, $d_G(v) \geqslant 2$. We note that all 2-vertices must be special by Claim 7.6. So if v is a special 2- or 3-vertex, then it must be adjacent to a vertex, say u, having degree at least 8 by Corollary 7.2. Actually, u is the base vertex of its incident flag. If u is not a pendant host, then we are done. Otherwise, assume that u is a pendant host having the unique neighbour u' in G^*. Note that $d_G(u') \geqslant 4$ by Claim 7.6 and Claim 7.7. Moreover, v cannot be a pendant host since otherwise G^* contains a complete graph K_2, violating Claim 7.3. Therefore, u' is a good 4^+-vertex.

Next suppose that v is a non-special 3-vertex. If v is a 3^{1^+}-vertex, then its 4^+-neighbour is just a good 4^+-vertex by Claim 7.7, and thus we are done. Now assume that v is a 3^0-vertex. By Claim 7.8, we are sure that v has at least two heavy neighbours each of which is adjacent to a good 6^+-vertex.

Finally suppose that v is a 4^+-vertex. If v is a pendant host, then its unique neighbour in G^*, denoted by u, must be a 4^+-vertex by Claim 7.6 and Claim 7.7, and it cannot be a pendant host by Claim 7.3, and therefore we are done. □

We have to present following two more useful lemmas.

Lemma 7.1. *If v is a 4^+-vertex in G, then $\omega^*(v) \geqslant \frac{1}{5}$ except in the following three cases:*
(A1) $d_G(v) = 4$ and $n_3(v) + ph(v) = 4$;
(A2) $d_G(v) \geqslant 6$, $f(v) = 5$ and $d^*(v) = 1$;
(A3) $d_G(v) \geqslant 6$, $f(v) = 5$ and $d^*(v) = n^*_{3^1}(v) = 2$.

Proof. Let v be a k-vertex with $k \geqslant 4$. If $k = 4$ and $n_3(v)+ph(v) \neq 4$, then it follows from (R2) and (R6) that $\omega^*(v) \geqslant 4 - \frac{16}{5} - 3 \times \frac{1}{5} = \frac{1}{5}$. If $k = 5$, then $\omega^*(v) \geqslant 5 - \frac{16}{5} - 5 \times \frac{1}{5} = \frac{4}{5}$

by (R3) and (R6). Now suppose that $k \geqslant 6$. By the discussion of Section 3.3.1, one may conclude that $\omega^*(v) \geqslant \frac{1}{5}$ if $f(v) \leqslant 4$, or $f(v) = 5$ and $d^*(v) \geqslant 3$, or $f(v) = 5$, $d^*(v) = 2$ and $n^*_{3^1}(v) \leqslant 1$. □

Lemma 7.2. *Let $v \in V(G)$ be a 6^+-vertex. Then $\omega^*(v) \geqslant \frac{2}{5}$ except in the following four cases:*
(B1) $f(v) = 2$ and $d^*(v) = 4$;
(B2) $f(v) = 3$, $d^*(v) = 3$ and $n^*_{3^1}(v) \geqslant 1$;
(B3) $f(v) = 4$ and $d^*(v) = 3$;
(B4) $f(v) = 5$ and $d^*(v) \in \{1, 2\}$.

Proof. It is trivial by the discharging argument of 6^+-vertices in Subsection 7.3.1. □

In what follows, let $z \in V(G)$ be a good 4^+-vertex by Claim 7.3. We have three cases in view of $d_G(z)$.

Case I: $d_G(z) = 5$.

Then $\omega^*(z) \geqslant 5 - \frac{16}{5} - 5 \times \frac{1}{5} = \frac{4}{5}$ by (R3) and (R6), and hence $\sum_{v \in V(G)} \omega^*(v) \geqslant \omega^*(z) \geqslant \frac{4}{5}$.

Case II: $d_G(z) \geqslant 6$.

Let $z_1, z_2, \ldots, z_{d^*(z)}$ denote all neighbours of z in G^*. Next, in order to show that $\sum_{v \in V(G)} \omega^*(v) \geqslant \frac{2}{5}$, by Lemma 7.2, we have to deal with four cases below:

Case (B1). $f(z) = 2$ and $d^*(z) = 4$. Then $\sigma(z) = \frac{8}{5}$. By Lemma 7.1, we assert that $\omega^*(z) \geqslant \frac{1}{5}$. Moreover, by Claim 7.13 (1) we know that v has at least either one 4^+-neighbour or one 3^{2^+}-neighbour, say z_1.

- First suppose that z_1 is a 4^+-neighbour. Clearly, z_1 cannot satisfy (A1) and (A3). If (A2) occurs, then z_1 is a pendant host, contradicting Claim 7.12. So by Lemma 7.1 we ensure that $\omega^*(z_1) \geqslant \frac{1}{5}$ and hence $\sum_{v \in V(G)} \omega^*(v) \geqslant \omega^*(z) + \omega^*(z_1) \geqslant \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$.
- Now suppose that z_1 is a 3^{2^+}-neighbour. If $n^*_{3^{2^+}}(v) \geqslant 2$, then $\omega^*(z) \geqslant \frac{8}{5} - 2 \times \frac{1}{5} - 2 \times \frac{2}{5} = \frac{2}{5}$ by (R4), meaning that $\sum_{v \in V(G)} \omega^*(v) \geqslant \omega^*(z) \geqslant \frac{2}{5}$. So next, assume that z_1 is the unique 3^{2^+}-neighbour of z. Denote by x_1, y_1 two neighbours of z_1 distinct from z. By Claim 7.13 (2), we see that z_1 is adjacent to at least two good 6^+-vertices, denoted by w_1, w_2, where $\{w_1, w_2\} \subset \{x_1, y_1, z\}$. It is obvious that w_1, w_2 cannot be vertices satisfying the cases of (A1)-(A3) of Lemma 7.1, and thus $\omega^*(w_1) \geqslant \frac{1}{5}$ and $\omega^*(w_2) \geqslant \frac{1}{5}$, implying that $\sum_{v \in V(G)} \omega^*(v) \geqslant \omega^*(w_1) + \omega^*(w_2) \geqslant \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$.

Case (B2). $f(z) = 3$, $d^*(z) = 3$ and $n^*_{3^1}(z) \geqslant 1$. By Lemma 7.1, we are sure that $\omega^*(z) \geqslant \frac{1}{5}$. W.l.o.g., assume that z_1 is a 3^1-neighbour of z. It follows from Claim 7.11 that z is adjacent to one good 6^+-vertex, say z_2. Clearly, z_2

cannot satisfy (A1) and (A2) since it is not a pendant host. Moreover, (A3) will not occur because z is not a 3^1-vertex. Hence, $\omega^*(z_2) \geqslant \frac{1}{5}$ by Lemma 7.1, and thus $\sum_{v\in V(G)} \omega^*(v) \geqslant \omega^*(z) + \omega^*(z_2) \geqslant \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$.

Case (B3). $f(z) = 4$ and $d^*(z) = 3$. Then $\sigma(z) = \frac{7}{5}$. By Lemma 7.1, we have that $\omega^*(z) \geqslant \frac{1}{5}$.

- If $n_{4^+}(z) \geqslant 1$, say z_1 is a 4^+-neighbour of z, then z_1 cannot be the cases of (A1) and (A3). If (A2) does not happen on z_1, then by Lemma 7.1, we obtain that $\omega^*(z_1) \geqslant \frac{1}{5}$ and therefore $\sum_{v\in V(G)} \omega^*(v) \geqslant \omega^*(z) + \omega^*(z_1) \geqslant \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$. Otherwise, assume that z_1 is a pendant host. By (R6), it gets charge $\frac{1}{5}$ from z, and thus $\omega^*(z) \geqslant \frac{7}{5} - \frac{1}{5} - 2 \times \frac{2}{5} = \frac{2}{5}$, implying that $\sum_{v\in V(G)} \omega^*(v) \geqslant \omega^*(z) \geqslant \frac{2}{5}$.
- If $n^*_{3^{2^+}}(z) \geqslant 1$, then by (R4.2), this 3^{2^+}-neighbour gets charge at most $\frac{1}{5}$ from z, and hence $\omega^*(z) \geqslant \frac{7}{5} - \frac{1}{5} - 2 \times \frac{2}{5} = \frac{2}{5}$, meaning that $\sum_{v\in V(G)} \omega^*(v) \geqslant \omega^*(z) \geqslant \frac{2}{5}$.
- Now suppose that $n^*_{3^1}(z) = 3$. If none of z_1, z_2 and z_3 is bad, then by (R4.1.2) and (R4.1.3) we affirm that z sends a charge of at most $3 \times \frac{3}{10} = \frac{9}{10}$ to all these 3-neighbours, and thus $\omega^*(z) \geqslant \frac{7}{5} - \frac{9}{10} = \frac{1}{2} > \frac{2}{5}$. It follows that $\sum_{v\in V(G)} \omega^*(v) \geqslant \omega^*(z) > \frac{2}{5}$. Otherwise, suppose w.l.o.g., that z_1 is bad. That is, z_1 is adjacent to exactly two 3^0-vertices, denoted by u_1 and u_2. By Claim 7.8, each u_i has two heavy neighbours which are adjacent to 6^+-vertices incident to at most four flags. For simplicity, let w_1 and w_2 denote these two 6^+-vertices. Notice that some w_i might be the same as z. But we can still ensure that one of w_1 and w_2 which is distinct from z, say w_1, cannot satisfy (A1) to (A3). Thus, $\omega^*(w_1) \geqslant \frac{1}{5}$ by Lemma 7.1 and, therefore, $\sum_{v\in V(G)} \omega^*(v) \geqslant \omega^*(z) + \omega^*(w_1) \geqslant \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$.

Case (B4). $f(z) = 5$ and $d^*(z) \in \{1, 2\}$. By assumption, z is not a pendant host, meaning that $d^*(z) = 2$, and then $\sigma(z) = \frac{4}{5}$. If each z_i gets a charge of at most $\frac{1}{5}$ from z, then $\omega^*(z) \geqslant \frac{4}{5} - 2 \times \frac{1}{5} = \frac{2}{5}$ and thus $\sum_{v\in V(G)} \omega^*(v) \geqslant \omega^*(z) \geqslant \frac{2}{5}$. Otherwise, by (R4.1.1) and (R4.1.2) we assume that each z_i is either weak or bad. So both z_1 and z_2 are 3^1-vertices. Let x_i and y_i denote other two neighbours of z_i distinct from z. By symmetry, assume that each x_i is a 3^0-vertex. By Claim 7.8, we know that for each $i \in \{1, 2\}$, at least two neighbours of x_i are heavy. Namely, there exists at least one 6^+-vertex, say w, such that $f(w) \leqslant 4$. Obviously, $w \neq z$ due to the fact that $f(z) = 5$. So by the discussion of (B1)-(B3), one may guarantee that $\sum_{v\in V(G)} \omega^*(v) \geqslant \omega^*(w) \geqslant \frac{2}{5}$.

Case III: $d_G(z) = 4$.

Then $\omega(z) = \frac{4}{5}$. Clearly, $f(z) = 0$ by Corollary 7.2. Let $N_G(z) = \{z_1, \ldots, z_4\}$. For $i \in \{1, \ldots, 4\}$, $d_G(z_i) \neq 1$ and $d_G(z_i) \neq 2$ due to Claim 7.6. Moreover, if z_i is a 5-vertex or a good 6^+-vertex, then we may go back to previous Case I and Case II. So in what follows, we assume that $n_3(z) + n_4(z) + ph(z) = 4$. There are two cases as follows:

- $n_3(z) + ph(z) \leqslant 3$. Say z_1 is a 4-vertex. Then $\omega^*(z) \geqslant \frac{4}{5} - 3 \times \frac{1}{5} = \frac{1}{5}$ by (R2) and (R6). Similarly, $\omega^*(z_1) \geqslant \frac{1}{5}$. Therefore, $\sum_{v \in V(G)} \omega^*(v) \geqslant \omega^*(z) + \omega^*(z_1) \geqslant \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$.

- $n_3(z) + ph(z) = 4$. If z is adjacent to at least two 3^{2^+}-vertices, say z_1 and z_2, then let z_i', z_i'' denote other two neighbours of z_i distinct to z for each $i \in \{1, 2\}$. W.l.o.g., assume that $d_G(z_1') \geqslant 4$ and $d_G(z_2') \geqslant 4$. Obviously, neither z_1' nor z_2' can be a pendant host by Claim 7.7, and thus we can suppose that both z_1' and z_2' are 4-vertices since otherwise we may reduce the argument to Case I and Case II. Again, applying (R2) and (R6), $\omega^*(z) \geqslant \frac{4}{5} - 2 \times \frac{1}{10} - 2 \times \frac{1}{5} = \frac{1}{5}$ and $\omega^*(z_i') \geqslant \frac{4}{5} - 3 \times \frac{1}{5} - \frac{1}{10} = \frac{1}{10}$ for both $i = 1, 2$. Therefore, $\sum_{v \in V(G)} \omega^*(v) \geqslant \omega^*(z) + \omega^*(z_1') + \omega^*(z_2') \geqslant \frac{1}{5} + 2 \times \frac{1}{10} = \frac{2}{5}$. Now suppose that z is adjacent to at most one 3^2-vertex. By Claim 7.14, G contains at least one good 5^+-vertex, and hence we can go back to Case I and Case II.

Therefore, we complete the proof of Theorem 7.1. □

7.4 Concluding remarks

We know that Borodin et al. [4] constructed a graph in $\mathcal{P}_6$ which has no (F_0, F_d)-partition, where d is a non-negative integer. Hence, this fact guarantees us that the subscript of F_1 in Corollary 7.1 cannot be further improved. Still, we suspect that the class of F_4 can be strengthened to F_3. To conclude this chapter, we would like to propose the following problem.

Question 7.1. Does every graph in $\mathcal{P}_6$ admit an (F_1, F_3)-partition?

Bibliography

[1] S. Arnborg, A. Proskurowski, and D. G. Corneil. Forbidden minors characterization of partial 3-trees. *Discrete Math.*, 80(1):1–19, 1990.

[2] O. V. Borodin. A proof of B. Grünbaum's conjecture on the acyclic 5-colorability of planar graphs. *Dokl. Akad. Nauk SSSR*, 231(1):18–20, 1976.

[3] O. V. Borodin and A. N. Glebov. On the partition of a planar graph of girth 5 into an empty and an acyclic subgraph. *Diskretn. Anal. Issled. Oper. Ser. 1*, 8(4):34–53, 2001.

[4] O. V. Borodin, A. O. Ivanova, M. Montassier, P. Ochem, and A. Raspaud. Vertex decompositions of sparse graphs into an edgeless subgraph and a subgraph of maximum degree at most k. *J. Graph Theory*, 65(2):83–93, 2010.

[5] O. V. Borodin and A. V. Kostochka. Defective 2-colorings of sparse graphs. *J. Combin. Theory Ser. B*, 104:72–80, 2014.

[6] G. G. Chappell, J. Gimbel, and C. Hartman. Thresholds for path colorings of planar graphs. In *Topics in discrete mathematics*, volume 26 of *Algorithms Combin.*, pages 435–454. Springer, Berlin, 2006.

[7] G. Chartrand and H. V. Kronk. The point-arboricity of planar graphs. *J. London Math. Soc.*, 44:612–616, 1969.

[8] M. Chen, A. Raspaud, and W. Wang. Vertex-arboricity of planar graphs without intersecting triangles. *European J. Combin.*, 33(5):905–923, 2012.

[9] M. Chen, A. Raspaud, W. Wang, and W. Yu. An (F_3, F_5)-partition of planar graphs with girth at least 5. *Discrete Math.*, 346(2):Paper No. 113216, 17, 2023.

[10] M. Chen, A. Raspaud, and W. Yu. An (F_1, F_4)-partition of graphs with low genus and girth at least 6. *J. Graph Theory*, 99(2):186–206, 2022.

[11] H. Choi, I. Choi, J. Jeong, and G. Suh. $(1, k)$-coloring of graphs with girth at least five on a surface. *J. Graph Theory*, 84(4):521–535, 2017.

[12] I. Choi and A. Raspaud. Planar graphs with girth at least 5 are $(3, 5)$-colorable. *Discrete Math.*, 338(4):661–667, 2015.

[13] I. Choi, G. Yu, and X. Zhang. Planar graphs with girth at least 5 are $(3, 4)$-colorable. *Discrete Math.*, 342(12):111577, 12, 2019.

[14] M. Chudnovsky, K. Edwards, K.-i. Kawarabayashi, and P. Seymour. Edge-colouring seven-regular planar graphs. *J. Combin. Theory Ser. B*, 115:276–302, 2015.

[15] M. Chudnovsky, K. Edwards, and P. Seymour. Edge-colouring eight-regular planar graphs. *J. Combin. Theory Ser. B*, 115:303–338, 2015.

[16] G. A. Dirac. On the structure of 5- and 6-chromatic abstract graphs. *J. Reine Angew. Math.*, 214(215):43–52, 1964.

[17] F. Dross, M. Montassier, and A. Pinlou. Partitioning a triangle-free planar graph into a forest and a forest of bounded degree. *European J. Combin.*, 66:81–94, 2017.

[18] Z. Dvořák, K.-i. Kawarabayashi, and D. Král'. Packing six T-joins in plane graphs. *J. Combin. Theory Ser. B*, 116:287–305, 2016.

[19] G. Fijavž, M. Juvan, B. Mohar, and R. Škrekovski. Planar graphs without cycles of specific lengths. *European J. Combin.*, 23(4):377–388, 2002.

[20] H. Gan and E. L. Johnson. Four problems on graphs with excluded minors. *Math. Programming*, 45(2, (Ser. B)):311–330, 1989.

[21] J. Geelen, B. Gerards, B. Reed, P. Seymour, and A. Vetta. On the odd-minor variant of Hadwiger's conjecture. *J. Combin. Theory Ser. B*, 99(1):20–29, 2009.

[22] A. M. H. Gerards. *Graphs and polyhedra. Binary spaces and cutting planes*, volume 73 of *CWI Tract*. Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, Amsterdam, 1990.

[23] B. Guenin. Packing t-joins and edge colouring in planar graphs. *manuscript*, 2003.

[24] B. Guenin. Packing odd circuit covers: A conjecture. *manuscript*, 2005.

[25] F. Harary. On the notion of balance of a signed graph. *Michigan Math. J.*, 2:143–146 (1955), 1953/54.

[26] F. Harary. On the measurement of structural balance. *Behavioral Sci.*, 4:316–323, 1959.

[27] F. Havet and J.-S. Sereni. Improper choosability of graphs and maximum average degree. *J. Graph Theory*, 52(3):181–199, 2006.

[28] F. Kardoš and J. Narboni. On the 4-color theorem for signed graphs. *European J. Combin.*, 91:Paper No. 103215, 8, 2021.

[29] W. Klostermeyer and C. Q. Zhang. $(2 + \epsilon)$-coloring of planar graphs with large odd-girth. *J. Graph Theory*, 33(2):109–119, 2000.

[30] N. Lacasse. Minimal and disjoint negation sets in signed graphs, 2020.

[31] X. Li, J. Liu, and J.-B. Lv. Every planar graph with girth at least 5 is $(1,9)$-colorable. *Discrete Math.*, 345(6):Paper No. 112818, 14, 2022.

[32] E. Máčajová, A. Raspaud, and M. Škoviera. The chromatic number of a signed graph. *Electron. J. Combin.*, 23(1):Paper 1.14, 10, 2016.

[33] M. Montassier and P. Ochem. Near-colorings: non-colorable graphs and NP-completeness. *Electron. J. Combin.*, 22(1):Paper 1.57, 13, 2015.

[34] R. Naserasr. Homomorphisms and edge-colourings of planar graphs. *J. Combin. Theory Ser. B*, 97(3):394–400, 2007.

[35] R. Naserasr, Y. Nigussie, and R. Škrekovski. Homomorphisms of triangle-free graphs without a K_5-minor. *Discrete Math.*, 309(18):5789–5798, 2009.

[36] R. Naserasr and L. A. Pham. Complex and homomorphic chromatic number of signed planar simple graphs. *Graphs Combin.*, 38(3):Paper No. 58, 22, 2022.

[37] R. Naserasr, E. Rollová, and E. Sopena. Homomorphisms of planar signed graphs to signed projective cubes. *Discrete Math. Theor. Comput. Sci.*, 15(3):1–11, 2013.

[38] R. Naserasr, E. Rollová, and E. Sopena. Homomorphisms of signed graphs. *J. Graph Theory*, 79(3):178–212, 2015.

[39] R. Naserasr, E. Sopena, and T. Zaslavsky. Homomorphisms of signed graphs: an update. *European J. Combin.*, 91:Paper No. 103222, 20, 2021.

[40] R. Naserasr and W. Yu. On the packing number of antibalanced signed simple planar graphs of negative girth at least 5. *Submitted*, 2022.

[41] R. Naserasr and W. Yu. Packing signatures in signed graphs. *Accepted for publication in SIAM J. Discrete Math.*, 2023.

[42] R. Naserasr and W. Yu. Separating signatures in planar signed graphs. *Accepted for publication in Discrete Appl. Math.*, 2023.

[43] K. S. Poh. On the linear vertex-arboricity of a planar graph. *J. Graph Theory*, 14(1):73–75, 1990.

[44] A. Raspaud and W. Wang. On the vertex-arboricity of planar graphs. *European J. Combin.*, 29(4):1064–1075, 2008.

[45] N. Robertson, D. Sanders, P. Seymour, and R. Thomas. The four-colour theorem. *J. Combin. Theory Ser. B*, 70(1):2–44, 1997.

[46] P. Seymour. *Matroids, Hypergraphs, and the Max,-Flow Min.-Cut Theorem.* PhD thesis, University of Oxford, 1975.

[47] P. G. Tait. Remarks on the colouring of maps. *Proc. Roy. Soc. Edinburgh.*, 10(729):501–503, 1880.

[48] R. Škrekovski. List improper colorings of planar graphs with prescribed girth. *Discrete Math.*, 214(1-3):221–233, 2000.

[49] W. Wang and K.-W. Lih. Choosability and edge choosability of planar graphs without five cycles. *Appl. Math. Lett.*, 15(5):561–565, 2002.

[50] T. Zaslavsky. Signed graph coloring. *Discrete Math.*, 39(2):215–228, 1982.

[51] T. Zaslavsky. Signed graphs. *Discrete Appl. Math.*, 4(1):47–74, 1982.